A STABILITY ANALYSIS FOR A FULLY NONLINEAR PARABOLIC PROBLEM IN DETONATION THEORY

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<u>Abstract :</u> This work is concerned with the the stability analysis of the constant stationary solution of the following fully nonlinear parabolic equation: $u_t + \frac{1}{2}u_x^2 = \mathbf{f}(cuu_{xx}) + \ln u$, $x \in (0,l)$ with $u_x(0,t) = u_x(l,t) = 0$, where $\mathbf{f}$ is a smooth function satisfying $\mathbf{f}(0) = 0, \mathbf{f}' > 0$ and $\mathbf{f}(\mathbb{R}) = \mathbb{R}$. In the case where $\mathbf{f}(s) = \ln\left[\frac{\exp(s)-1}{s}\right]$, this equation represents the evolution of the perturbations of the Zeldovich-von Neuman-Doering square wave occuring during a detonation in a duct. We first study the stationary solutions and reveal a bifurcation phenomenon.

Then, by formulating the problem as an abstract equation defined on a suitable Banach space, we are able to use the extension to fully nonlinear problems of the classical geometric theory for semilinear parabolic equations. In this way, we prove that the equilibrium point $u_0 = 1$ is unstable. Moreover, a more careful description of a special class of initial conditions for which u_0 is stable.

1. INTRODUCTION

Let us consider the following fully nonlinear problem:

$$
\begin{cases}
u_t + \dfrac{1}{2}\,u_x^2 \;=\; \mathbf{f}(\,c\,u\,u_{xx}\,) \;+\; \ln\,u, & x \in (0\,,l),\ t > 0 \\[2em]
u_x\,(0\,,t) \;=\; 0,\ u_x\,(l\,,t) \;=\; 0, & t > 0 \\[2em]
u\,(x\,,0) \;=\; U\,(x) \;>\; 0, & x \in (0\,,l)
\end{cases}
\tag{1}
$$

where the derivatives are denoted by indexes, c and l are positive real constants, $\mathbf{f}$ is a real smooth function of one variable satisfying $\mathbf{f}(0) = 0$, $\mathbf{f}' > 0$ and $\mathbf{f}(\mathbb{R}) = \mathbb{R}$.

This paper is concerned with the stability of the unique constant stationary solution of Problem (1), namely $u_0 \equiv 1$. Such a physical problem occurs in detonation theory (see Buckmaster and Ludford [4−5]) for the particular choice where $\mathbf{f}(s) = \ln\left[\dfrac{\exp(s) - 1}{s}\right]$. From the mathematical point of view, the fully nonlinear character of Problem (1) requires a very specific treatment; in fact the already classical geometric theory of semilinear parabolic equations (see Henry [13]) must be replaced by a recently generalised version by Da Prato-Lunardi[9], and Lunardi[15−16] to fully nonlinear parabolic problems. They proved the existence of the invariant manifolds of a stationary solution in the same manner as Henry, provided that the derivative of the operator computed at this stationary solution, say L_0, generates an analytic semigroup in interpolation spaces between the domain of its iterate. We refer to Da Prato-Grisvard[8] for the formal functional analysis framework. The stability of the trivial solution $u_0 \equiv 1$ is investigated using their results. Precisely, we state that this equilibrium state is unstable by mean of a linearised stability principle. Moreover, a more careful description of the stable, unstable and center-unstable manifolds[14] leads to the determination of a special class of initial conditions for which u_0 may be stable.

The paper is organised as follows: in section 2 we present the origin of the Problem (1) in the special case of a detonation in a duct. Section 3 is devoted to the existence of stationary solution of Problem (1) as well as the related bifurcation phenomenon:

in our detonation problem, we can observe, by the use of a numerical proof, a global bifurcation phenomenon. Section 4 deals with the local existence and the instability of the trivial stationary solution u_0. Section 5 is devoted to the non singular (no null eigenvalue) case of saddle point configuration. We present the existence of the stable and unstable manifolds and we get an approximation of the stable manifold by formulating the question of the stability of the trivial solution as a problem in finding a root of a differential mapping defined in suitable Banach spaces and by applying the Implicit Function Theorem. We point out that this method being constructive, it provides a numerical scheme to approximate the stable manifold[2−12].

Let us mention that various aspects of this problem have been previously studied by two of the authors and coworkers:

- in [2], they formally studied the stability of the trivial stationary solution and they performed a numerical approximation of the stable manifold in the non singular case,

- in [1], they have used a semilinear approximation of Problem (1) to completely study the stability of the trivial stationary solution and to show off the effect of the logarithmic term on the singular behaviour of the evolutive solution, namely the quenching phenomenon.

The quenching phenomenon for the fully nonlinear Problem (1) is investigated in a forthcoming paper by Galaktionov, Gerbi and Vasquez[11].

2. FROM THE PHYSICAL PROBLEM TO THE MATHEMATICAL FORMULATION

Detonations waves are for the most part unstable[10], and it is important to under-stand the origins and the consequences of the instability. Since activation energy is a valuable tool in flame theory (low Mach number combustion)[20], it is natural to apply it to detonations (high Mach number phenomenon) in the same way.

Consider a detonation wave propagating down a channel of length L. The steady detonation structure is characterised by an induction zone of length δ, following an hydrodynamic shock wave, and introducing a vigorous reaction in which heat release occurs. We refer to Fickett[10] for a more complete description. Suppose that the viscous effects are negligible and that the chemical reaction is reduced to one gas burning to give a product, then the governing equations are the compressible reactive Euler equations:

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho\, u) = 0 \\[2em]
\rho \dfrac{D\, u}{D\, t} + \operatorname{grad}(p) = 0 \\[2em]
\rho \dfrac{D\, H}{D\, t} = \dfrac{D\, p}{D\, t} + Q\,\Omega \\[2em]
\rho \dfrac{D\, Y}{D\, t} = -\,\Omega
\end{cases}
$$

in which ρ stands for the density of the gas, u its velocity, p its pressure, H its enthalpy, Y the mass fraction of the product, Q the heat of the chemical reaction and Ω the reaction rate.

For the sake of simplicity the gas is supposed to be perfect. The chemical reaction is described by a one step Arrhénius law; then the preceeding system of conservation laws is completed by the following state equations :

$$\Omega = B\, Y \,\exp(-\frac{E}{RT})\;;\; H = \frac{\gamma}{\gamma - 1}\frac{p}{\rho},$$

where $\gamma = \dfrac{C_p}{C_v}$ is the massic heat ratio, E the activation energy and R the universal gas constant.

In the limit of the high activation energy, the detonation structure is reduced to the famous Zeldovich-von Neuman-Doering square wave, denoted by ZND[10]. But the instability of plane detonation waves gives rise to transverse propagation of secondary shock waves across the face of the main shock. Taking as non dimensionalised energy $\theta = \dfrac{(E\, C_p)}{(R\, u_f^2)}$ where u_f is the longitudinal speed of the shock, and searching the disturbances of the main shock in the wave length scale $y \approx \delta\,\sqrt{\theta}$, for a time scale $t \approx \dfrac{\delta\,\theta}{u_f}$, the shock position is defined as:

$$x_{shock} = x_{ZND} + \delta\, h\,(\frac{y}{\delta\,\sqrt{\theta}},\, \frac{t\, u_f}{\delta\,\theta}).$$

Writing the Rankine-Hugoniot relations, developing all the variables in the high energy asymptotics and supposing the wall perfectly reflecting, Buckmaster and Ludford[4–5] get the following evolution equation for $g = (1 + \dfrac{h}{K})$ where K is a positive constant:

$$
\begin{cases}
g_t + \dfrac{1}{2}\, g_y^2 = \ln\big[\dfrac{\exp(c\, g\, g_{yy}) - 1}{c\, g_{yy}}\big], & y \in (0,\, l),\ t > 0 \\[3mm]
g_y(0,\, t) = 0,\ g_y(l,\, t) = 0, & t > 0 \\[3mm]
g(y,\, 0) = G(y) > 0, & y \in (0,\, l)
\end{cases}
\tag{2}
$$

where c is a nondimensionalised positive constant representing the chemical properties and l is a nondimensionalised positive constant representing the geometrical properties. Typically, for a detonation whose overdrive coefficient is D = 1.2, a perfect gas of massic heat ratio $\gamma = 1.2$ and a nondimensionalised heat of reaction

$\bar{Q} = \dfrac{Q}{R T_f} = 50$, we have $c = 0.268$.

Due to the change of unknowns between g and h, the non perturbed ZND wave is represented by the stationary constant solution $g_0 \equiv 1$. A natural extension of Problem (2) is obtained by considering $\mathbf{f}$ to be a real function belonging to $C^\infty(\mathbb{R})$ satisfying $\mathbf{f}(0) = 0$, $\mathbf{f'} > 0$ and $\mathbf{f}(\mathbb{R}) = \mathbb{R}$ and by studying the following model problem:

$$
\begin{cases}
u_t + \dfrac{1}{2}\, u_x^2 = \mathbf{f}(\, c\, u\, u_{xx}\,) + \ln u, & x \in (0,\, l),\ t > 0 \\[2em]
u_x\,(0\,,\ t) = 0,\ u_x\,(l\,,\ t) = 0, & t > 0 \\[2em]
u\,(x\,,\ 0) = U\,(x) > 0, & x \in (0\,,\ l)
\end{cases}
\tag{3}
$$

We naturally recover Problem (2) by taking $\mathbf{f}(s) = \ln\big[\dfrac{\exp(s) - 1}{s}\big]$.

At this stage, we can remark that Problem (3) is a fully nonlinear parabolic problem since the highest space derivative u_{xx} is contained in the nonlinearity and $\mathbf{f'} > 0$. It admits only one constant stationary solution $u_0 \equiv 1$, since $\mathbf{f}(0) = 0$.

In the following section, we will briefly describe the stationary solutions of Problem (3) and will show off a global bifurcation phenomenon.

3. STATIONARY SOLUTIONS

The stationary solutions satisfy the nonlinear one dimensional elliptic problem:

$$
\begin{cases}
u'' = \dfrac{1}{c\, u}\, \mathcal{F}\,(\dfrac{u'^2}{2} - \ln u) \\[2em]
u'\,(0) = 0\,,\ u'\,(l) = 0
\end{cases}
\tag{4}
$$

where $\mathcal{F}$ is the reciprocal function of $\mathbf{f}$ i.e. $\mathcal{F} \circ \mathbf{f} = \mathbf{f} \circ \mathcal{F} = I$; it satisfies $\mathcal{F} \in C^\infty(\mathbb{R})$, $\mathcal{F}(0) = 0$, $\mathcal{F}' > 0$, and $\mathcal{F}(\mathbb{R}) = \mathbb{R}$.

In [12] a complete study of the stationary solutions is carried out. First a local bifurcation phenomenon is shown via classical Crandall-Rabinowitz results[7]. Moreover a global bifurcation analysis is performed by introducing the so-called time-map as in Chafee-Infante[6], Smoller-Wasserman[19] and recently Schaaf[18]. We recall here the principal results on the global bifurcation phenomenon (see [12] for detailed proofs).

For $m > 0$, we shall study the initial value problem:

$$
\begin{cases}
u'' = \dfrac{1}{c\,u}\,\mathcal{F}\left(\dfrac{u'^2}{2} - \ln u\right) \\[4mm]
u(0) = m\,, \ u'(0) = 0
\end{cases}
\tag{5}
$$

whose solution will be denoted by $u(.\,;\,m)$. Let $J(m)$ be the maximal interval $(0\,,\,x^*)$ such that $u(x\,;\,m) > 0$, for every x in $J(m)$. We can now define the time-map T, as follows:

$$
D(T) = \{\,m > 0\,,\ m \neq 1/\,\exists\,x \in J(m)\,,\ u'(x\,;\,m) = 0\,\}\,,
$$
$$
\forall\,m \in D(T)\,,\ T(m) = \mathrm{Min}\,\{x \in J(m)\,,\ u'(x\,;\,m) = 0\}.
$$

We introduce a function F defined by:

$$
\forall\,s < \mathbf{f}(c)\,,\ F(s) = \int_0^s \frac{\mathcal{F}(x)}{\mathcal{F}(x) - c}\,dx\,.
$$

From the hypothesis on $\mathbf{f}$, one can observe that F is twice continuously differentiable, concave, negative and $F(0) = 0$. Therefore we can define a function g by:

$$
\forall\,s < \mathbf{f}(c)\,,\ s \neq 0\,,\ s\,g(s) < 0 \text{ and } F(s) = F(g(s)).
$$

Thus we get the following caracterisation of the domain:

Proposition 1:

$$
D(T) = (m^*\,,\,+\infty) \text{ where } m^* = \exp(-\,\mathbf{f}(c)).
$$

proof:

We perform the change of unknowns:

$r = u'$

$s = \dfrac{u'^2}{2} - \ln u$

which is a diffeomorphism from $\mathbb{R}^+ \times \mathbb{R}$ to $\mathbb{R}^2$.

We obtain the following differential system:

$$
\begin{cases}
r' = \dfrac{1}{c}\,\exp(s - \dfrac{r^2}{2})\,\mathcal{F}(s) & r(0) = 0 \\[4mm]
s' = \dfrac{r}{c}\,\exp(s - \dfrac{r^2}{2})\,(\mathcal{F}(s) - c) & s(0) = -\ln m
\end{cases}
\tag{6}
$$

We seek $m > 0$ such that there exists $t > 0$ verifying $r(t) = 0$. It is clear that $s(t) = \mathbf{f}(c)$ is a separatrix for the Problem (6), i.e:

i) if $s(0) > \mathbf{f}(c)$ then for every $t > 0$, $s(t) > \mathbf{f}(c)$, and $r(t) > 0$.

ii) if $s(0) = \mathbf{f}(c)$ then for every $t > 0$, $s(t) = \mathbf{f}(c)$, and $r(t) > 0$.

iii) if $s(0) < \mathbf{f}(c)$ then for every $t > 0$, $s(t) < \mathbf{f}(c)$.

Therefore if $s(0) \geq \mathbf{f}(c)$, i.e. $m \leq m^*$, then $m \notin D(T)$. Thus $D(T) \subset (m^*, +\infty)$.
Let $m \in (m^*, +\infty)$. As $s(0) < \mathbf{f}(c)$, for every $t > 0$, $F(s(t))$ is well defined.

It is now clear that Problem (6) has the first integral:

$$
\forall\, t > 0, \qquad F(s(t)) - \dfrac{r(t)^2}{2} = \text{Cte} = F(s(0)) = F(-\ln m).
\tag{7}
$$

From this first integral, the point $(s = g(s(0))\,,\, r = 0)$ is on the trajectory issued from $(s(0)\,,\, 0)$.

Therefore there exists $t > 0$, such $r(t) = 0$. Thus $(m^*, +\infty) \subset D(T)$ $\bullet$

With this notation, we obtained the following caracterisation of the solution $u(.\,;\, m)$ and the following explicit formula for the time-map:

Proposition 2:

For every $m \in (m^*, +\infty)$, $m \neq 1$, the function $u(.\,; m)$ is periodic with smallest half-period $T(m)$. Moreover we have:

$$T(m) = \text{sign}(-\ln(m)) \frac{c}{\sqrt{2}} \int_{g(-\ln(m))}^{-\ln(m)} \frac{\exp(F(s) - F(-\ln(m)) - s)}{\sqrt{F(s) - F(-\ln(m))}} \frac{ds}{c - f(s)}$$

proof:

From the first integral (7) and the symetry, a phase plane study of Problem (6) shows that if (s, r) is on a trajectory then the points $(s, -r)$ and $(g(s), r)$ are also on this trajectory.

But since for every $s < \mathbf{f}(c)$, $s \neq 0$, $g(g(s)) = s$, the trajectories are closed graphs. Thus r is periodic and as $r = u'$, u is periodic too. Taking into account the symmetry, it is clear that $T(m)$ is the smallest half-period of $u(.\,; m)$.

Moreover, writing:

$$T(m) = \int_0^{T(m)} 1 \, dt$$

and performing the change of variable: $s = s(t)$, $ds = s'(t)dt = \frac{r}{c} \exp(s - \frac{r^2}{2})(\mathcal{F}(s) - c)dt$, we get the explicit formula of T given in proposition 2.•

As F is concave, non positive and increasing on $[0, \mathbf{f}(c))$, we perform the Chafee-Infante's change of variables: $F(s) = F(x)\,y^2$, where $y \in [0, 1]$. With this tool, we can differentiate the preceeding formula with respect to m and we obtained the following result on the variation of T:

Proposition 3:

Let j be the real function defined by: for every $s \in (-\infty, \mathbf{f}(c))$,
$$j(s) = \frac{\exp(F(s) - s)}{\mathcal{F}(s)^3}(\mathcal{F}(s)^2 + 2F(s)(c - \mathcal{F}(s))(\mathcal{F}'(s) + \mathcal{F}(s))).$$

Let us denote by $l_0 = \pi \sqrt{\dfrac{c}{\mathcal{F}'(0)}}$.

If for every $s \in (0, \mathbf{f}(c))$, $j(s) - j(g(s)) \geq 0$, then T is a one-to-one twicely differ-entiable mapping from $(m^*, 1]$ on $[l_0, +\infty)$.

The proof of proposition 3 follows the same method that Schaaf's one[18] and need only some technical computations. We leave these computations to the reader. Then, we obtain the following global bifurcation phenomenon:

Theorem 1 (global bifurcation phenomenon):

For $k \in \mathbb{N}$, we denote by $U_k = \{ u \in C^2([0, l]), u - 1 \text{ has } k \text{ zeros in } [0, l] \}$.

i) If $l < \text{Min}\{T(m), m \in (m^*, +\infty)\}$, Problem (4) has only one solution: the trivial one.

ii) If $l \geq \text{Min}\{T(m), m \in (m^*, +\infty)\}$, Problem (4) has at least two non trivial solutions.

iii) We suppose that for every $s \in (0, \mathbf{f}(c))$, $j(s) - j(g(s)) \geq 0$, and $l \geq l_0$. Set $k = E[l/l_0]$ the integer part of l/l_0.

Problem (4) has exactly $2k$ non trivial solutions where two belong to U_1, two to U_2 ,..., two to U_k.

Set $u^+(x; l)$ the solution belonging to U_1 such that $u(0) > 1$ and $u^-(x; l)$ the solution belonging to U_1 such that $u(0) < 1$. The function $l \in [l_0, +\infty) \to u^+(0; l)$ is increasing while the function $l \in [l_0, +\infty) \to u^-(0; l)$ is decreasing.

proof:

It rests only on the fact that $u(.; m)$ is periodic with smallest half-period $T(m)$. Thus, it suffices that the length of the interval l contains an integer time the half-period $T(m)$, and because of the symmetry due to the function g, two solutions exist with the same half-period: the one which satisfies $u(0; m) > 1$ and the other $u(0; m) < 1$.

If $\mathbf{f}$ is the function defined by $\mathbf{f}(s) = \ln\left[\dfrac{\exp(s) - 1}{s}\right]$, in order to prove that the inequality: $\forall s \in (0, \mathbf{f}(c))$, $j(s) - j(g(s)) \geq 0$ is true, we use numerical evaluations

of the functions $\mathcal{F}$, F, j and g defined above; we thus can state that we have the global bifurcation phenomenon of stationary solutions. In figures 1 and 2, we present the time-map and the global bifurcation phenomenon.
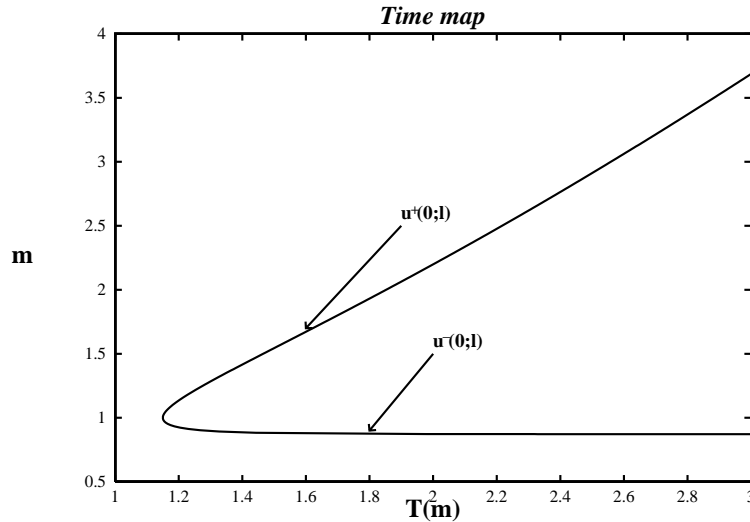

Figure 1: Time-map

4. LOCAL EXISTENCE AND LINEARISED STABILITY PRINCIPLE.

In order to use abstract results on local existence and stability for stationary solutions of a fully nonlinear parabolic problem, [9−15] we shall interpret Problem (1) as an evolution equation in a suitable Banach space.

Remark:

As the goal of this paper is to investigate the stability of the constant stationary solution $U_0 \equiv 1$, for $\rho > 0$, we regularise the singular part of Problem (1), namely the logarithmic term, as a function $\ln_\rho$ belonging to $C^\infty(\mathbb{R})$ whose value is $\ln s$ if $s > \rho$. For simplicity in the notations, we still denote by $\ln$ the regularised function $\ln_\rho$.

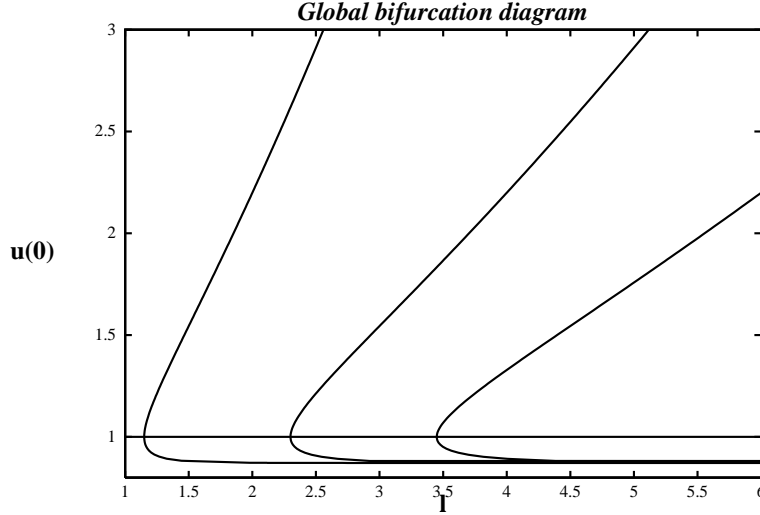As in the recent paper of Lunardi[16], we consider the space of Hölder continuous

Figure 2: Global bifurcation phenomenon

functions: for $\theta \in (0\,,\,1/2)$, and $n \in \mathbb{N}$,

$$C^{n+2\theta}([0\,,\,l]) = \{v \in C^n([0\,,\,l])\,,\, \sup_{\substack{x,y \in [0,l] \\ x \neq y}} \frac{|u^{(n)}(x) - u^{(n)}(y)|}{|x-y|^{2\theta}} < \infty\}$$

This space is a Banach space endowed with the norm:

$$\text{for } u \in C^{n+2\theta}([0\,,\,l])\,,\, \|u\|_{n+2\theta} = \sum_{k=0}^{n} \operatorname*{Max}_{x \in [0,l]} \mid u^k(x) \mid + \sup_{\substack{x,y \in [0,l] \\ x \neq y}} \frac{\mid u^{(n)}(x) - u^{(n)}(y) \mid}{\mid x - y \mid^{2\theta}}.$$

The open ball of center w and radius R from $C^{n+2\theta}([0\,,\,l])$ is denoted by $B_{n+2\theta}(w\,,\,R)$. Set $X = \{v \in C^{2+2\theta}([0\,,\,l])\,,\, v'(0) = v'(l)\}$. We introduce a mapping defined on X whose value belongs to $C^{2\theta}([0\,,\,l])$ as follows:

$$F: \qquad X \longrightarrow C^{2\theta}([0\,,\,l])$$

$$u \longmapsto -\tfrac{1}{2}\,u_x^2 + \mathbf{f}(c\,u\,u_{xx}) + \ln u.$$

As we supposed that $\mathbf{f}$ belongs to $C^\infty(\mathbb{R})$, it is clear that F is indefinitely differentiable on X. Setting $u(t) = u(.\,,\,t)$, we write Problem (1) as the infinite dimensional dynamical system:

$$\begin{cases} u_t = F(u) \\ u(0) = U \end{cases} \tag{8}$$

We denote by L_0 the derivative of F at the equilibrium point u_0 defined by:

$$\text{for every } v \in X, \ L_0(v) = \mathbf{f}'(0)\,c\,v_{xx} + v. \tag{9}$$

Its spectrum $\sigma(L_0)$ consists in a countable number of simple eigenvalues $\lambda_k = 1 - \dfrac{k^2\,\pi^2\,c\,\mathbf{f}'(0)}{l^2}$, $k \in \mathbb{N}$, with corresponding eigenvectors $w_k(x) = \cos(\dfrac{k\,\pi\,x}{l})$. The operator L_0 is a sectorial operator and generates an analytic semigroup $\exp(L_0\,t)$ on $C^{2\theta}([0\,,\,l])$. Therefore the local existence result of Da Prato-Lunardi[9] (prop.2.1) and Lunardi[16] (thm.2.1) can be applied:

Theorem 2 (local existence):

For every $T > 0$, there exists $R(T)$, such that, for every $U \in X$ verifying $\|U - u_0\|_{2+2\theta} \leq R(T)$, there exists a unique solution of Problem (1), $u \in \mathcal{C}((0\,,\,T]\,;\,X) \bigcap \mathcal{C}^\theta((0\,,\,T]\,;\,C^2([0\,,\,l])) \bigcap \mathcal{C}^1((0\,,\,T]\,;\,C^{2\theta}([0\,,\,l]))$ •

The linearised stability principle for semilinear parabolic equation, Henry[13], has been generalised by Da Prato-Lunardi[9] (thm.2.3) and recently improved by Lunardi[16] (thm.2.5) to the fully nonlinear parabolic equations. As for every $l > 0$, $\lambda_0 = 1$ is an eigenvalue of L_0, this principle states that the equilibrium point u_0 is unstable.

Theorem 3 (instability):

Let $\omega \in (0\,,\,1)$. There exists $r > 0$, $\rho > 0$ such that: for every $U \in B_{2+2\theta}(u_0\,,\,r)$, there exists a unique $u \in \mathcal{C}((-\infty\,,\,0]\,;\,X) \bigcap \mathcal{C}^1((-\infty\,,\,0]\,;\,C^{2\theta}([0\,,\,l]))$ backward solution of Problem (1) and for every $t < 0$, $\|u(t) - u_0\|_{2+2\theta} \leq \rho\,\exp(\omega\,t)$ •

Since there always exists a strictly positive eigenvalue, two cases have to be distinguished: the non singular case when there is no null eigenvalue, it is the saddle point configuration and the singular case when a null eigenvalue exists, it is the center manifold configuration.

5. SADDLE POINT CONFIGURATION

In this section, we suppose that no null eigenvalue exists. We first state the existence of the stable and unstable manifolds and secondly we study the local behaviour of the stable manifold and we present an approximation of it.

Let us split the spectrum of L_0 in: $\sigma_+ = \{\lambda \in \sigma(L_0), \lambda > 0\} = \{\lambda_j\}_{j=0,..,n}$ and,

$$\sigma_- = \{\lambda \in \sigma(L_0), \lambda < 0\} = \{\lambda_j\}_{j=n+1,n+2,..}.$$

Let us denote by $E^u = \operatorname{span}(w_i, i = 0, ..., n)$, the unstable space and P_u the projection of X on E^u. Set $P_s = I - P_u$ and $E^s = P_s X$; clearly $X = E^s \oplus E^u$.

5.1 Existence of stable and unstable manifolds.

The saddle point theorem for semilinear parabolic equations Henry[13] stays true in this functional analysis framework for fully nonlinear problems, Da Prato-Lunardi[8] (thm.2.4) and Lunardi[16] (thm.2.8).

Theorem 4 (saddle point configuration)

Let $\omega \in (0, -\lambda_{n+1})$.

There exists $r > 0$, $\rho > 0$, and two unique Lipschitz continuous mappings, differentiable at u_0 defined by:

$h:$ $\qquad B^u_{2+2\theta}(u_0, r) \subset E^u \longrightarrow E^s$,

$k:$ $\qquad B^s_{2+2\theta}(u_0, r) \subset E^s \longrightarrow E^u$,

and two manifolds:

$W^u = \{u + h(u), u \in B^u_{2+2\theta}(u_0, r) \subset E^u\}$, $W^s = \{u + k(u), u \in B^s_{2+2\theta}(u_0, r) \subset E^s\}$,

such that:

i) for every $U \in W^s$, there exists a unique $U \in \mathcal{C}((0, +\infty)\,;\, X) \bigcap \mathcal{C}^1((0, +\infty)\,;\, C^{2\theta}([0, l]))$ solution of Problem (1) and for every $t > 0$, $\|u(t) - u_0\|_{2+2\theta} \leq \rho \exp(-\omega t)$.

Conversely, if U is such that $\|P_s U\|_{2+2\theta} \leq r$, and if the solution u of Problem (1) verifies: $u \in \mathcal{C}((0, +\infty)\,;\, X) \bigcap \mathcal{C}^1((0, +\infty)\,;\, C^{2\theta}([0, l]))$ and for every $t > 0$, $\|u(t) - u_0\|_{2+2\theta} \leq \rho \exp(-\omega t)$ then $u \in W^s$.

ii) for every $U \in W^u$, there exists a unique $u \in \mathcal{C}((-\infty, 0]\,;\, X) \bigcap \mathcal{C}^1((-\infty, 0]\,;\, C^{2\theta}([0, l]))$ backward solution of Problem (1) and for every $t < 0$, $\|u(t) - u_0\|_{2+2\theta} \leq \rho \exp(\omega t)$.

Conversely if U is such that $\|P_u U\|_{2+2\theta} \leq r$, and if the backward solution of Problem (1) u verifies: $u \in \mathcal{C}((-\infty, 0]\,;\, X) \bigcap \mathcal{C}^1((-\infty, 0]\,;\, C^{\theta}([0, l]))$ and for every $t < 0$, $\|u(t) - u_0\|_{2+2\theta} \leq \rho \exp(\omega t)$ then $U \in W^u$.

iii) $h(u_0) = k(u_0) = u_0$ and $h'(u_0) = k'(u_0) = 0$.

u_0 is called a saddle point, W^s is the stable manifold at u_0 and W^u is the unstable manifold at u_0 •.

5.2 Approximation of the stable manifold.

In this section, we are interested in the local behaviour of the stable manifold. Our method is complementary to the one of Da Prato and Lunardi because we will construct the stable manifold by using the Implicit Function Theorem formulated in suitable Banach spaces whereas they proved its local existence by using a fixed point theorem which is non constructive. Thus a numerical method to approximate the stable manifold can be carried out. For this purpose, we will follow the method used by D.H.Sattinger to investigate the stability of waves of nonlinear parabolic systems[17]. This method has already been used in fully nonlinear context to investigate the stability of travelling front[3].

For $\varepsilon \in \mathbb{R}$, set $u = u_0 + \varepsilon v$, and decompose the initial condition $U = u_0 + \varepsilon \xi^s + \varepsilon^2 \xi^u$, where ξ^s belongs to E^s and ξ^u belongs to E^u are to be

determined in order to get the solution belongs to W^s.

By a Taylor serie expansion of F up to the second order, Problem (8) becomes:

$$
\begin{cases}
v_t = L_0(v) + \dfrac{\varepsilon}{2} H_0(v, v) + \varepsilon^2 R(\varepsilon; v) \\[2em]
v(0) = \xi^s + \varepsilon \xi^u
\end{cases}
\tag{10}
$$

H_0 is the Hessian of F at u_0, defined by: for every $v, w \in X$,

$$
H_0(v, w) = \mathbf{f}''(0)c^2 v_{xx}w_{xx} + \mathbf{f}'(0)c(wv_{xx} + vw_{xx}) - v_x w_x - vw
\tag{11}
$$

and $R(\varepsilon; v)$ is the integral remainder defined by: for every $v \in X$, for every $\varepsilon \in \mathbb{R}$,

$$
R(\varepsilon; v) = \frac{1}{2} \int_0^1 (1 - \sigma)^2 F^{(3)}(u_0 + \sigma \varepsilon v)(v, v, v) \, d\sigma
\tag{12}
$$

Denoting by L_s (resp. L_u) the restriction of L_0 to E^s (resp. E^u), $\varphi^s = P_s v$ and $\varepsilon \varphi^u = P_u v$, when projecting Problem (8) on E^s and E^u, we get:

$$
\begin{cases}
\varphi_t^s = L_s(\varphi^s) + \dfrac{\varepsilon}{2} P_s H_0(\varphi^s + \varepsilon \varphi^u, \varphi^s + \varepsilon \varphi^u) + \varepsilon^2 P_s R(\varepsilon; \varphi^s + \varepsilon \varphi^u) \\[2em]
\varphi^s(t = 0) = \xi^s
\end{cases}
\tag{13}
$$

and,

$$
\begin{cases}
\varphi_t^u = L_u(\varphi^u) + \dfrac{1}{2} P_u H_0(\varphi^s + \varepsilon \varphi^u, \varphi^s + \varepsilon \varphi^u) + \varepsilon P_u R(\varepsilon; \varphi^s + \varepsilon \varphi^u) \\[2em]
\varphi^u(t = 0) = \xi^u
\end{cases}
\tag{14}
$$

As L_u generates a semi-group $\exp(L_u t)$, φ^u is expressed as:

$$
\begin{aligned}
\varphi^u(t) = \exp(L_u t)[\xi^u + \int_0^1 \exp(-L_u \sigma)\{\frac{1}{2} P_u H_0(\varphi^s + \varepsilon \varphi^u, \varphi^s + \varepsilon \varphi^u) \\
+ \varepsilon P_u R(\varepsilon; \varphi^s + \varepsilon \varphi^u)\} \, d\sigma]
\end{aligned}
\tag{15}
$$

Therefore as we want to follow an orbit entering the equilibrium point u_0, i.e. $\varphi^u \to 0$ as $t \to \infty$, ξ^u is formally expressed as:

$$
\begin{aligned}
\xi^u = -\int_0^{+\infty} \exp(-L_u\sigma)\{\frac{1}{2}P_u H_0(\varphi^s + \varepsilon\varphi^u, \varphi^s + \varepsilon\varphi^u) \\
+\varepsilon P_u R(\varepsilon; \varphi^s + \varepsilon\varphi^u)\}\, d\sigma
\end{aligned}
\tag{16}
$$

In the same way, φ^u is formally expressed as:

$$
\begin{aligned}
\varphi^u(t) = -\int_t^{+\infty} \exp(-L_u(\sigma - t))\{\frac{1}{2}P_u H_0(\varphi^s + \varepsilon\varphi^u, \varphi^s + \varepsilon\varphi^u) \\
+\varepsilon P_u R(\varepsilon; \varphi^s + \varepsilon\varphi^u)\}\, d\sigma
\end{aligned}
\tag{17}
$$

In order to use the Implicit Function Theorem, let us define the following Banach spaces. Let $\omega \in (0, -\lambda_{n+1})$ and $i \in \mathbb{N}$. Set $C_{\omega,i}$ the space of functions u belonging to $\mathcal{C}((0,+\infty); C^{i+2\theta}([0,l]))$ such that the function $t \to \exp(\omega t)\, u(t)$ belongs to $L^\infty((0,+\infty); C^{i+2\theta}([0,l]))$.

This space endowed with the norm, $\|v\|_{\omega,i} = \sup\limits_{t>0} \|\exp(\omega t)\, v(t)\|_{i+2\theta}$ is a Banach space.

For $w \in C_{\omega,0}$, consider the problem:

$$
\begin{cases}
\varphi_t^s = L_s(\varphi^s) + P_s\, w(t) \\
\\
\varphi^s(t=0) = 0
\end{cases}
\tag{18}
$$

In [16], prop.2.4, it is shown that there exists $C > 0$, independant of w and $\varphi^s \in P_s C_{\omega,2}$ solution of Problem (18), such that:

$$
\|\varphi^s\|_{\omega,2} \leq C\, \|w\|_{\omega,0}
$$

Therefore, if we denote by K_s the transformation from $C_{\omega,0}$ to $P_s C_{\omega,2}$ defined by $w \to \varphi^s$, the preceeding inequality shows that K_s is a bounded linear operator. In

order to check that the right hand side of Problem (13) belongs to $C_{\omega,0}$, we need the two following lemmas:

Lemma 1:

H_0 is $\mathcal{C}^\infty$ from $C_{\omega,2} \times C_{\omega,2}$ to $C_{\omega,0}$.

proof:

As H_0 is a bilinear form from $X \times X$ to $C^{2\theta}$, we have:

$\exists C > 0, \forall v \in X, \forall w \in X, \|H_0(v,w)\|_{2\theta} \leq C \|v\|_{2+2\theta} \|w\|_{2+2\theta}$.

Thus, for every $v \in C_{\omega,2}$, and $w \in C_{\omega,2}$, $H_0(v,w) \in C_{\omega,0}$.

As H_0 is bilinear, H_0 is indefinitely differentiable $\bullet$

Lemma 2:

R is $\mathcal{C}^\infty$ from $\mathbb{R} \times C_{\omega,2}$ to $C_{\omega,0}$.

proof:

As F is indefinitely differentiable on X, there exists $M > 0$, such that:

$\forall v \in X, \forall \varepsilon \in \mathbb{R}, \|R(\varepsilon;v)\|_{2\theta} \leq M \|v\|_{2+2\theta}^3$.

Thus, for every $v \in C_{\omega,2}$, and for every $\varepsilon \in \mathbb{R}$, $R(\varepsilon;v) \in C_{\omega,0}$.

From the property of F, it is clear that R is an indefinitely differentiable map from $\mathbb{R} \times C_{\omega,2}$ to $C_{\omega,0}$ $\bullet$

We thus can formally express the solution φ^s of Problem (13) as:

$$\varphi^s(t) = \exp(L_s t)\xi^s + \varepsilon K_s(\frac{1}{2}P_s H_0(\varphi^s + \varepsilon\varphi^u, \varphi^s + \varepsilon\varphi^u) + \varepsilon P_s R(\varepsilon; \varphi^s + \varepsilon\varphi^u)) \quad (19)$$

Equations (16), (17) and (19) may be collect as follows.

For $(\varphi^s, \varphi^u, \xi^u; \varepsilon) \in P_s C_{\omega,2} \times P_u C_{\omega,2} \times E^u \times \mathbb{R}$, we define:

$$\mathcal{G}_1(\varphi^s, \varphi^u, \xi^u; \varepsilon) = \varphi^s(t) - \exp(L_s t)\xi^s - \varepsilon K_s(\frac{1}{2}P_s H_0(\varphi^s + \varepsilon\varphi^u, \varphi^s + \varepsilon\varphi^u)$$
$$+\varepsilon P_s R(\varepsilon; \varphi^s + \varepsilon\varphi^u)),$$

$$\mathcal{G}_2(\varphi^s, \varphi^u, \xi^u; \varepsilon) = \xi^u + \int_0^{+\infty} \exp(-L_u\sigma)\{\frac{1}{2}P_u H_0(\varphi^s + \varepsilon\varphi^u, \varphi^s + \varepsilon\varphi^u)$$
$$+\varepsilon P_u R(\varepsilon; \varphi^s + \varepsilon\varphi^u)\}\, d\sigma,$$

$$\mathcal{G}_3(\varphi^s, \varphi^u, \xi^u; \varepsilon) = \varphi^u(t) + \int_t^{+\infty} \exp(-L_u(\sigma - t))\{\frac{1}{2}P_u H_0(\varphi^s + \varepsilon\varphi^u, \varphi^s + \varepsilon\varphi^u)$$
$$+\varepsilon P_u R(\varepsilon; \varphi^s + \varepsilon\varphi^u)\}\, d\sigma$$

Now setting $\mathcal{G}(\varphi^s, \varphi^u, \xi^u; \varepsilon) = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$, Equations (16), (17) and (19) may be written in compact form:

$$\mathcal{G}(\varphi^s, \varphi^u, \xi^u; \varepsilon) = 0. \tag{20}$$

For small ε, we wish to construct solutions of Equation (20) in the form $(\varphi^s(\varepsilon), \varphi^u(\varepsilon), \xi^u(\varepsilon); \varepsilon)$.

As all eigenvalues of L_s are negative less that λ_{n+1}, it follows from lemma 1 and lemma 2 that $\mathcal{G}_1$ is $\mathcal{C}^\infty$ from $P_s C_{\omega,2} \times P_u C_{\omega,2} \times E^u \times \mathbb{R}$ to $P_s C_{\omega,2}$. Samely, $\mathcal{G}_2$ is $\mathcal{C}^\infty$ from $P_s C_{\omega,2} \times P_u C_{\omega,2} \times E^u \times \mathbb{R}$ to E^u. To prove that $\mathcal{G}$ is $\mathcal{C}^\infty$, we need the following lemma:

<u>Lemma 3:</u>
$\mathcal{G}_3$ is $\mathcal{C}^\infty$ from $P_s C_{\omega,2} \times P_u C_{\omega,2} \times E^u \times \mathbb{R}$ to $P_u C_{\omega,2}$

<u>proof:</u>
From lemma 1 and lemma 2 we get:
$\forall v \in C_{\omega,2}, \forall \varepsilon \in \mathbb{R}, H_0(v, v) + \varepsilon R(\varepsilon; v) \in C_{\omega,0}$.
Thus:
$\forall t > 0, \forall v \in C_{\omega,2}, \|H_0(v, v) + \varepsilon R(\varepsilon; v)\|_{2\theta} \le \exp(-\omega t)\|H_0(v, v) + \varepsilon R(\varepsilon; v)\|_{\omega,0}$
As the linear operator $-L_u$ generates an analytic semi-group on $C^{2\theta}$ whose largest eigenvalue is $-\lambda_n < 0$, and as P_u is a projector, we get $\forall t > 0, \forall \sigma \ge t, \forall v \in C_{\omega,2}$,

$\| \exp(-L_u(\sigma-t))(P_u H_0(v,v)+\varepsilon P_u R(\varepsilon;v)\|_{2+2\theta} \leq \exp(-\lambda_n(\sigma-t)\|H_0(v,v)+\varepsilon R(\varepsilon;v)\|_{2\theta}$

Therefore, there exists $M(\omega,\lambda_n) > 0$ such that

$\forall t > 0, \forall(\varphi^s, \varphi^u, \varepsilon) \in P_s C_{\omega,2} \times P_u C_{\omega,2} \times \mathbb{R},$

$\| \int_t^{+\infty} \exp(-L_u(\sigma-t))\{\frac{1}{2}P_u H_0(\varphi^s+\varepsilon\varphi^u, \varphi^s+\varepsilon\varphi^u) + \varepsilon P_u R(\varepsilon; \varphi^s+\varepsilon\varphi^u)\} \, d\sigma\|_{2+2\theta}$

$\leq M \exp(-\omega t)\|H_0(\varphi^s+\varepsilon\varphi^u, \varphi^s+\varepsilon\varphi^u) + \varepsilon R(\varepsilon; \varphi^s+\varepsilon\varphi^u)\|_{\omega,0}$

From lemma 1 and lemma 2, we finally get:

$\forall(\varphi^s, \varphi^u, \xi^u; \varepsilon) \in P_s C_{\omega,2} \times P_u C_{\omega,2} \times E^u \times \mathbb{R}, \mathcal{G}_3(\varphi^s, \varphi^u, \xi^u; \varepsilon) \in P_u C_{\omega,2}.$

Moreover as H_0 and R are $\mathcal{C}^\infty$, $\mathcal{G}_3$ is $\mathcal{C}^\infty$ ∎

From our hypothesis on the spectrum of L_0, when $\varepsilon = 0$, we may take:

$$\varphi^s(0) = \varphi_0^s = \exp(L_s t)\xi^s \in P_s C_{\omega,2} \tag{21}$$

$$\varphi^u(0) = \varphi_0^u = - \int_t^{+\infty} \exp(-L_u(\sigma-t))\frac{1}{2}P_u H_0(\varphi_0^s, \varphi_0^s)d\sigma \in P_u C_{\omega,2} \tag{22}$$

$$\xi^u(0) = \xi_0^u = -\frac{1}{2}\int_t^{+\infty} \exp(-L_u\sigma)P_u H_0(\varphi_0^s, \varphi_0^s)d\sigma \in E^u. \tag{23}$$

Moreover it is clear that $(\varphi_0^s, \varphi_0^u, \xi_0^u; 0)$ verifies: $\mathcal{G}(\varphi_0^s, \varphi_0^u, \xi_0^u; 0) = 0.$

We may thus apply the Implicit Function Theorem to the mapping $\mathcal{G}$. To this end, we compute the derivative of $\mathcal{G}$ at $(\varphi_0^s, \varphi_0^u, \xi_0^u)$, $\mathcal{G}_0'$. By construction:

$$\mathcal{G}_0' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the only extra diagonal term which can be non-zero is the partial derivative of $P_u H_0(\varphi^s+\varepsilon\varphi^u, \varphi^s+\varepsilon\varphi^u)$ with respect to φ^s. But,

$$\frac{\partial(P_u H_0(\varphi^s+\varepsilon\varphi^u, \varphi^s+\varepsilon\varphi^u))}{\partial\varphi^s}(\varphi_0^s, \varphi_0^u, \xi_0^u; 0) = P_u H_0(1, \varphi^s) = P_u(L_s\varphi^s - 2\varphi^s) = 0.$$

This operator being clearly invertible, we state the following result:

Theorem 5:

There exists $\varepsilon_0 > 0$ and three unique $\mathcal{C}^\infty$ functions:

$\varphi_s : (-\varepsilon_0, \varepsilon_0) \to P_s C_{\omega,2}$, $\varphi_u : (-\varepsilon_0, \varepsilon_0) \to P_u C_{\omega,2}$, $\xi^u : (-\varepsilon_0, \varepsilon_0) \to E^u$, satisfying

i) $\forall \varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\mathcal{G}(\varphi^s(\varepsilon), \varphi^u(\varepsilon), \xi^u(\varepsilon); \varepsilon) = 0$.

ii) $(\varphi^s(0), \varphi^u(0), \xi^u(0)) = (\varphi_0^s, \varphi_0^u, \xi_0^u)$.

Therefore for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, for every $\xi^s \in E^s$, if $U = u_0 + \varepsilon\xi^s + \varepsilon^2\xi^u(\varepsilon)$, then there exists a unique $u \in \mathcal{C}((0, +\infty) ; X) \bigcap \mathcal{C}^1((0, +\infty) ; C^{2\theta}([0,l]))$ solution of Problem (1) and for every $t > 0$, $\|u(t) - u_0\|_{2+2\theta} \leq \rho \exp(-\omega t)$ ∎

5.3 Numerical approximation of the stable manifold

For small ε, we get easily a first order approximation of the stable manifold by taking $(\varphi^s(\varepsilon), \varphi^u(\varepsilon), \xi^u(\varepsilon)) \approx (\varphi_0^s, \varphi_0^u, \xi_0^u)$.

Clearly the vector space span $(w_i, i = n+1, n+2 ...)$ is included in E^s. Therefore, to build a numerical approximation of the stable manifold, one can take ξ^s as a finite sum of these eigenvectors.

Let $p \in \mathbb{N}$, $p > n+1$ and let $(\alpha_k^s)_{k=n+1,..,p}$ be a sequence of $p-n$ real numbers. Set $\xi^s = \sum_{k=n+1}^{p} \alpha_k^s w_k$ and consider the decomposition of ξ_0^u on the basis of E^u,

$\xi_0^u = \sum_{j=0}^{n} \beta_j^u w_j$.

Denoting for $j = 0, ..n$, P_j the projector of X on the eigenspace spanned by w_j, we get the analytical formula[2,12]:

$$\text{for } j = 0, .., n, \quad \beta_j^u = \frac{1}{2} \sum_{\substack{k=n+1 \\ m=n+1}}^{p} \alpha_k^s \alpha_m^s P_j H_0(w_k, w_m) \frac{1}{\lambda_k + \lambda_m - \lambda_j} \qquad (24)$$

The preceeding formula gives us an approximation of the stable manifold as follows:

Theorem 6 (numerical approximation of the stable manifold):

Let $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, small be given.

If $U(x) = 1 + \varepsilon \sum_{k=n+1}^{p} \alpha_k^s \cos(\frac{k\pi x}{l}) + \varepsilon^2 \sum_{j=0}^{n} \beta_j^u \cos(\frac{j\pi x}{l})$, with β_j^u given by the formula (24) then there exists a unique global solution of Problem (8) converging to u_0 as t tends to infinity, the rate of convergence being exponential $\blacksquare$

Application to the detonation problem:

In our physical motivation, $\mathbf{f}$ is the function defined by $\mathbf{f}(s) = \ln \left[\dfrac{\exp(s) - 1}{s} \right]$. Taking $l < l_0$, such that 1 is the only strictly positive eigenvalue, i.e. $n = 0$, we denote by $r = l/l_0$. Choosing $p = 1$ and, for the sake of simplicity in the computation of β_0^u, supposing $\alpha_1^s = 1$, one can take as initial condition, $U(x) = 1 + \varepsilon \cos(\dfrac{\pi x}{l}) + \varepsilon^2 \beta_0^u$, with:

$$\beta_0^u = \frac{1}{2 - r^2} \left[\frac{-1}{12r^2} + \frac{1+c}{2c} + \frac{r^2}{4} \right] \tag{25}$$

A numerical method for solving Problem (1) with this initial condition has been used in [2], and a good agreement between the numerical computation of β_0^u and the analytic formula (25) has been observed.

6. CENTER MANIFOLD CONFIGURATION

In this section we suppose that there exists a null eigenvalue and for the sake of simplicity in the algebra, we suppose that this eigenvalue is the second thus $l = l_0$. We split the spectrum of L_0 in $\sigma_+ = \{\lambda_0\}$, $\sigma_0 = \{\lambda_1 = 0\}$ and $\sigma_- = \{\lambda_j\}_{j=2,3,\ldots}$. We denote by $E^u = \text{span}(u_0)$ the unstable space, $E^c = \text{span}(w_1)$ the center space and P_u (resp. P_c) the projection of X on E^u (resp. E^c). Set $P_s = I - P_u - P_c$ and $E^s = P_s X$.

The existence of a center-unstable manifold in the semilinear case, Kelley[14], Henry[10]

(thm.6.2.1), has been adapted to the fully nonlinear case by Da Prato-Lunardi[9] (thm.3.1) and Lunardi[16] (thm.3.2).

Theorem 7 (center-unstable manifold):

There exists $r > 0$, and a Lipschitz continuous mapping, indefinitely differentiable at u_0:

$h: \qquad B_{2+2\theta}^{c,u}(u_0, r) \subset E^c \times E^u \to E^s$, such that:

i) for every $U \in X$, verifying $(P_c U, P_u U) \in B_{2+2\theta}^{c,u}(u_0, r)$ and $P_s U = h(P_c U, P_u U)$, there exists a unique $u \in \mathcal{C}((0, +\infty); X) \bigcap \mathcal{C}^1((0, +\infty); C^{2\theta}[0, l])$ solution of Problem (1) and for every $t > 0$, $P_s u(t) = h(P_c u(t), P_u u(t))$.

ii) $h(P_c u_0, P_u u_0) = u_0$ and $h'(P_c u_0, P_u u_0) = 0$.

The manifold $W^{c,u} = \{P_c u + P_u u + h(P_c u, P_u u), (P_c u, P_u u) \in B_{2+2\theta}^{c,u}(u_0, r) \subset E^c \times E^u\}$ is called the center-unstable invariant manifold $\bullet$

As usual, to investigate the stability of the equilibrium point u_0 with respect to the center-unstable manifold, we have to project Problem (8) on the three spaces E^s, E^u, E^c, and perform a Taylor serie expansion of F up to the third order. Setting $u = u_0 + v$, Problem (8) becomes:

$$
\begin{cases}
v_t = L_0(v) + \dfrac{1}{2} H_0(v, v) + \dfrac{1}{6} N_0(v, v, v) + R(v) \\
\\
v(0) = V
\end{cases}
\tag{26}
$$

where V is small and $N_0(v, v, v)$ is the third derivative of F at u_0, defined by:

$$
\text{for every } v \in X, \ N_0(v, v, v) = \mathbf{f}^{(3)}(0)(cv_{xx})^3 + 6\mathbf{f}''(0)c^2 v(v_{xx})^2 + \frac{v^3}{2}
\tag{27}
$$

To simplify the notation, for every $v \in X$, we denote by:

$G(v) = \frac{1}{2} H_0(v, v) + \frac{1}{6} N_0(v, v, v) + R(v)$.

Denoting by L_s (resp. L_c, L_u) the restriction of L_0 to E^s (resp. E^c, E^u), $\varphi^s = P_s v$, $\varphi^u = P_u v$ and $\varphi^c = P_c v$, when projecting Problem (8) on E^s, E^c and E^u we get:

$$
\begin{cases}
\varphi_t^s = L_s(\varphi^s) + P_s G(v) \\[2em]
\varphi_t^u = \varphi^u + P_u G(v) \\[2em]
\varphi_t^c = P_c G(v)
\end{cases}
\tag{28}
$$

But by Theorem 7, $\varphi^s = h(\varphi^c, \varphi^u)$. Moreover h is determinated by [9]:

$$
h(\varphi^c, \varphi^u) = \int_{-\infty}^{0} \exp(-\sigma L_s) P_s G(h(\varphi^c, \varphi^u) + \varphi^c + \varphi^u) \, d\sigma
\tag{29}
$$

In the semilinear case, Kelley[14] proved that studying the stability of u_0 with respect to the center-unstable manifold is equivalent to study the stability of the 2-dimensional dynamical system:

$$
\begin{cases}
\varphi_t^u = \varphi^u + P_u G(h(\varphi^c, \varphi^u) + \varphi^c + \varphi^u) \\[2em]
\varphi_t^c = P_c G(h(\varphi^c, \varphi^u) + \varphi^c + \varphi^u)
\end{cases}
\tag{30}
$$

Lunardi[16] proved that this property stays true in the fully nonlinear case. Therefore, we shall look for a local center manifold for Problem (30) whose existence has been established by Kelley[14]. More precisely, there exists an open ball $B_{2+2\theta}^c(u_0, r_c) \subset E^c$ and a Lipschitz continuous mapping indefinitely differentiable at u_0, $\tau : B_{2+2\theta}^c(u_0, r_c) \subset E^c \to E^u$, satisfying $\tau(u_0) = \tau'(u_0) = 0$ determined by:

$$
\varphi^u(t=0) = \int_0^{\infty} \exp(-\sigma) P_u G(h(\varphi^c, \tau(\varphi^c) + \varphi^c + \tau(\varphi^c)) \, d\sigma = \tau(\varphi^c(t=0))
\tag{31}
$$

This local center manifold is defined by: $\Sigma = \{\varphi^c + \tau(\varphi^c),\ \varphi^c \in B^c_{2+2\theta}(u_0, r_c) \subset E^c\}$

Finally, inserting $\tau(\varphi^c)$ in Problem (30), we are led to the 1-dimensional ordinary differential equation:

$$\varphi^c_t = P_c G(h(\varphi^c, \tau(\varphi^c) + \varphi^c + \tau(\varphi^c) \equiv \mathcal{P}(\varphi^c) \tag{32}$$

As E^c is a one dimensional vector space, we parametrise it as $\varphi^c = s(t)\, w_1$. Taking into account that $\tau(u_0) = \tau'(u_0) = 0$ and $h'(u_0, u_0) = 0$, we have:

$$\tau(s\, w_1) = \frac{s^2}{2}\, \tau''(0)(w_1, w_1) + \mathrm{o}(s^3) \tag{33}$$

$$h(s\, w_1, \tau(s\, w_1)) = \frac{s^2}{2}\, h''_1(0,0)(w_1, w_1) + \mathrm{o}(s^3) \tag{34}$$

Where h''_1 stands for the second derivative of h with respect to the "first variable".

Substituting these expressions into the operator $\mathcal{P}$ defined by Equation (32) we get:

$$\mathcal{P}(\varphi^c) = \frac{s^2}{2}H_0(w_1, w_1) + \frac{s^3}{2}H_0(w_1, h''_1(0,0)(w_1, w_1)) + \frac{s^3}{2}H_0(w_1, \tau''(0,0)(w_1, w_1))$$
$$+\frac{s^3}{6}N_0(w_1, w_1, w_1) + \mathrm{o}(s^4).$$

Thus Equation (32) may be simplified as:

$$s_t w_1 = \frac{s^2}{2}P_c H_0(w_1, w_1) + \frac{s^3}{2}P_c H_0(w_1, h''_1(0,0)(w_1, w_1))$$
$$+\frac{s^3}{2}P_c H_0(w_1, \tau''(0,0)(w_1, w_1)) + \frac{s^3}{6}P_c N_0(w_1, w_1, w_1) + \mathrm{o}(s^4) \tag{35}$$

After some computations, we find $P_c H_0(w_1, w_1) = 0$.

But from Equations (29), (31), we obtain:

$$h''_1(0,0)(w_1, w_1) = \int_{-\infty}^{0} \exp(-\sigma L_s) Q H_0(w_1, w_1)\, d\sigma \tag{36}$$

and,

$$\tau''(0,0)(w_1, w_1) = -\int_{0}^{\infty} \exp(-\sigma I) P_u H_0(w_1, w_1)\, d\sigma = P_u H_0(w_1, w_1) \tag{37}$$

Symplifying by w_1 in Equation (35), we finally obtain:

$$s_t = \frac{s^3}{2\,l_0}(A_1 + A_2 + A_3) + o(s^4) \tag{38}$$

with $\quad A_1 = < H_0(w_1\,, \int_{-\infty}^{0} \exp(-\sigma L_s)P_s H_0(w_1, w_1)\,d\sigma)\,,\; w_1 >,$

$\qquad A_2 = < H_0(w_1\,,\; P_u H_0(w_1, w_1))\,,\; w_1 >,$

$\qquad A_3 = \frac{1}{3}\; < N_0(w_1, w_1, w_1)\,,\; w_1 >,$

where $< .\,, . >$ stands for the scalar product of $L^2(0, l)$.

After some algebraic computations, we get:

$$A_1 + A_2 + A3 = \pi\,\frac{\sqrt{c\,\mathbf{f}'(0)}\,A}{48\,c^2\,\mathbf{f}'(0)^4},\; \text{with,}$$

$$A = -2\mathbf{f}'(0)^2 + 24c\mathbf{f}'(0)^3 + 93c^2\mathbf{f}'(0)^4 + 2c\mathbf{f}'(0)\mathbf{f}''(0) - 6c^2\mathbf{f}'(0)^2\mathbf{f}''(0) + 4c^2\mathbf{f}''(0)^2$$
$$-6\mathbf{f}'(0)\mathbf{f}^{(3)}(0)$$

Thus $A_1 + A_2 + A3$ has the same sign of A. Since the stability of u_0 with respect to the center-unstable manifold is equivalent of the stability of the zero state for the ordinary differential equation (38), we can now conclude this study by:

Theorem 8 (stability with respect to the center-unstable manifold):

Suppose $l = l_0$.

There exists $r > 0$ and a Lipschitz continuous mapping, indefinitely differentiable at u_0:

$h: \qquad B^{c,u}_{2+2\theta}(u_0, r) \subset E^c \times E^u \to E^s$, such that we are in one of the two following cases:

i) Suppose that A is strictly negative (stability).

Then there exists $\rho > 0$ and $\omega > 0$, such that:

for every $U \in X$, verifying $(P_c U, P_u U) \in B_{2+2\theta}^{c,u}(u_0, r)$ and $P_s U = h(P_c U, P_u U)$, there exists a unique $u \in \mathcal{C}([0, +\infty) \, ; \, X) \bigcap \mathcal{C}^1([0, +\infty); C^{2\theta}([0, l]))$ solution of Problem (1) and for every $t > 0$, $P_s u(t) = h(P_c u(t), P_u u(t))$, and $\|u(t) - u_0\|_{2+2\theta} \le \rho \exp(-\omega t)$.

ii) Suppose that A is strictly positive (instability).

Then there exists $\rho > 0$ and $\omega > 0$, such that:

for every $U \in X$, verifying $(P_c U, P_u U) \in B_{2+2\theta}^{c,u}(u_0, r)$ and $P_s U = h(P_c U, P_u U)$, there exists a unique $u \in \mathcal{C}((-\infty, 0] \, ; \, X) \bigcap \mathcal{C}^1((-\infty, 0]; C^{2\theta}([0, l]))$ backward solution of Problem (1) and for every $t < 0$, $P_s u(t) = h(P_c u(t), P_u u(t))$, and $\|u(t) - u_0\|_{2+2\theta} \le \rho \exp(\omega t) \bullet$

Remarks:

1) There exists R_c such that:

$B_{2+2\theta}^c(u_0, R_c) \subset B_{2+2\theta}^c(u_0, r_c) \subset E^c$ and $B_{2+2\theta}^c(u_0, R_c) \times \tau(B_{2+2\theta}^c(u_0, r_c)) \subset B_{2+2\theta}^{c,u}(u_0, r)$, and the manifold $\Sigma' = \{\varphi^c + \tau(\varphi^c) + h(\varphi^c, \tau(\varphi^c)), \ \varphi^c \in B_{2+2\theta}(0, R_c)\}$ is an invariant manifold for Problem (1). We have the following stability diagram:
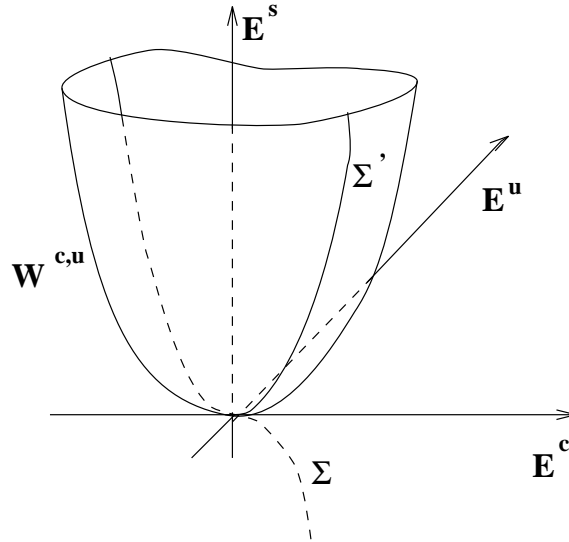

Figure 3: Stability diagram

2) Observe that from Equations (33) and (34), any $w \in \Sigma'$ has the local expansion:

$$w = sw_1 + \frac{s^2}{2}\tau''(0)(w_1, w_1) + \frac{s^2}{2}h_1''(0,0)(w_1, w_1) + o(s^3).$$

After computations we get:

$$w(x) = s\cos(\frac{\pi x}{l_0}) + \frac{s^2}{2}(-\frac{3}{2} - \frac{1}{2c\mathbf{f}'(0)} + \frac{\mathbf{f}''(0)}{2\mathbf{f}'(0)^2}) + \frac{s^2}{6}(-\frac{3}{4} + \frac{1}{4c\mathbf{f}'(0)} + \frac{\mathbf{f}''(0)}{2\mathbf{f}'(0)^2})\cos(\frac{2\pi x}{l_0})$$
$$+ o(s^3)$$

3) Recall that in our detonation problem, $\mathbf{f}$ is the function defined by $\mathbf{f}(s) = \ln\left[\frac{\exp(s) - 1}{s}\right]$.
Thus:

$$A = -\frac{1}{2} + c\frac{37}{12} + c^2\frac{823}{144}$$

Let c^* be the positive root of $A = 0$, i.e. $c^* = \dfrac{-444 + 36\sqrt{355}}{1646} \approx 0.103057$

The stability of u_0 occurs if $0 < c < c^*$ and the instability occurs if $c > c^*$. Thus in

our test problem $c = 0.268$, the ZND wave is not only unstable but also unstable

with respect to the center-unstable manifold.

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