SIAM J. APPL. MATH.
Vol. 61, No. 4, pp. 1253–1285

QUENCHING FOR A ONE-DIMENSIONAL FULLY NONLINEAR PARABOLIC EQUATION IN DETONATION THEORY*

VICTOR A. GALAKTIONOV†, STÉPHANE GERBI‡, AND JUAN L. VAZQUEZ§

Abstract. We study and describe the quenching phenomenon for the fully nonlinear parabolic equation

$$u_t + \frac{1}{2}(u_x)^2 = f(c\,u\,u_{xx}) + \ln u, \quad x \in (0,l),\ t > 0,$$

which for $f(s) = \ln[(e^s - 1)/s]$ represents the evolution of the perturbations of the Zel'dovich–von Neuman–Doering square wave occurring during a detonation in a duct. In the general case, the function $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies the parabolicity condition $f'(s) > 0$ in $\mathbb{R}$, c and l are positive constants, and we impose Neumann boundary conditions $u_x(0,t) = u_x(l,t) = 0$ for $t > 0$ and take initial data $u(x,0) > 0$ with inverse bell-shaped form.

The phenomenon of quenching is characterized by the existence of a finite time T at which the solution u ceases to exist as a classical solution because $\min_x u(x,t) \to 0$ as $t \to T$; then the equation degenerates and forms a singularity at the level $u = 0$, due to the presence of the logarithmic zero-order term.

We first exhibit conditions on f and $u(x,0)$ which imply the presence of this type of singularity. Next we derive estimates on u, u_{xx} in order to study the behavior of the profile in the neighborhood of the time T. We then find the asymptotic scaling factors, which are universal, and the asymptotic profile which is given in the rescaled coordinates by a parabola with a free constant to adjust. For this purpose we use the theory of stability of ω-limit sets of infinite-dimensional dynamical systems under asymptotically small perturbations. In this problem the perturbation is singular but exponentially vanishing as $t \to T$. Finally, we prove that the present model does not admit any extension beyond the singularity, i.e., for $t > T$.

Key words. detonation, fully nonlinear equation, quenching, asymptotic behavior, singular perturbations, Hamilton–Jacobi equation

AMS subject classifications. 35K55, 35K65

PII. S0036139998341928

1. Introduction. Mathematical formulation of the detonation problem.

This work is concerned with the study of the following initial boundary-value problem:

$$\begin{cases} u_t + \dfrac{1}{2}(u_x)^2 = f(c\,u\,u_{xx}) + \ln u\,, & x \in (0,l)\,,\ t > 0, \\ u_x(0,t) = u_x(l,t) = 0\,, & t > 0, \\ u(x,0) = u_0(x) > 0\,, & x \in (0,l). \end{cases} \tag{1.1}$$

This problem is proposed by Buckmaster [5], Buckmaster and Ludford [7], and Buckmaster, Dold, and Schmidt-Lainé [6] in the study of detonation waves. In these works

*Received by the editors July 7, 1998; accepted for publication (in revised form) May 26, 2000; published electronically November 10, 2000.
http://www.siam.org/journals/siap/61-4/34192.html

†Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK, and Keldysh Institute of Applied Mathematics, Miusskaya Sq. 4, 125047 Moscow, Russia (vag@maths.bath.ac.uk). The research of this author was supported by EU TMR contract FMRX-CT98-0201.

‡Laboratoire de Mathématiques, Université de Savoie, GM^3, Campus Scientifique, 73376 Le Bourget du Lac Cedex, France (gerbi@univ-savoie.fr).

§Departamento de Matemáticas, Universidad Autonoma de Madrid, 28049 Madrid, Spain (juanluis.vazquez@uam.es). The research of this author was supported by EU TMR contract FMRX-CT98-0201 and was also partially funded by DGES grant PB94-0153.

c and l are real positive constants and f is given by the formula

$$f(s) = \ln\left(\frac{e^s - 1}{s}\right). \tag{1.2}$$

The main physical and mathematical feature of the problem is the occurrence of the phenomenon called *quenching*, which is formulated as follows: the solution $u(x,t)$ of problem (1.1) exists in the classical sense and is positive up to a finite time $T > 0$ such that

$$\min_{x\in(0,l)} u(x,t) \to 0 \qquad \text{as } t \to T. \tag{1.3}$$

At the *quenching time* $t = T$ the classical solution ceases to exist because a singularity occurs in the right-hand side of the equation. The description of this singularity, i.e., the behavior of the solution for $t \approx T$, is of great importance for the understanding of the detonation process.

For the sake of mathematical generality, we will consider in this paper a more general real, convex, and smooth function $f : \mathbb{R} \to \mathbb{R}$ satisfying the parabolicity condition $f'(s) > 0$ in $\mathbb{R}$ and normalized with $f(0) = 0$, plus the limits $f'(s) \to \lambda_1 > 0$ as $s \to +\infty$ and $|s|f'(s) \to \lambda_2$ as $s \to -\infty$, where λ_1, λ_2 are positive constants. Finally, we impose a technical but essential condition (6.2). All these assumptions are satisfied by (1.2), which is a convex function with $f'(0) = 1/2$, $f'(\infty) = 1$, $f'(-\infty) = 0$. Consequently, and for simplicity, we take $\lambda_1 = 1$ throughout. The initial data $u_0(x)$ are assumed to be positive and inversely bell-shaped in form.

Let us review the formulation of the detonation problem in the form (1.1). Detonation waves are for the most part unstable [9], and it is important to understand the origins and the consequences of the instability. Since activation energy is a valuable tool in flame theory (low Mach number combustion) [33], it is natural to apply it also to detonation which is a high Mach number phenomenon.

Consider a detonation wave propagating down a channel of length L. The steady detonation structure is characterized by an induction zone of length δ, following a hydrodynamic shock wave, and introducing a vigorous reaction in which heat release occurs. We refer to Fickett and Davis [9] for a more complete description. Suppose that the viscous effects are negligible and that the chemical reaction is reduced to one gas burning to give one product. Then the governing equations are the compressible reactive Euler equations

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\, u) = 0,\ \ \rho\frac{du}{dt} + \operatorname{grad}(p) = 0,\ \ \rho\frac{dH}{dt} = \frac{dp}{dt} + Q\,\Omega,\ \ \rho\frac{dY}{dt} = -\Omega,$$

in which ρ stands for the density of the gas, u its velocity, p its pressure, H its enthalpy, Y the mass fraction of the product, Q the heat of the chemical reaction, and Ω the reaction rate. For the sake of simplicity the gas is supposed to be perfect. The chemical reaction is described by a one-step Arrhenius law; then the preceding system of conservation laws is completed by the following state equations:

$$\Omega = BY\, e^{-E/RT},\ H = \frac{\gamma}{\gamma-1}\frac{p}{\rho}\,,$$

where $\gamma = C_p/C_v$ is the massic heat ratio, E the activation energy, and R the universal gas constant.

In the limit of the high activation energy, the detonation structure is reduced to the famous Zel'dovich–von Neuman–Doering square wave, denoted by ZND [9]. This stationary wave is a step in the (x, T) plane from T_{burn} to T_{cold} ($T_{\text{burn}} \gg T_{\text{cold}}$) located at the point x_{ZND} traveling at the speed u. The instability of plane detonation waves gives rise to transverse propagation of secondary shock waves across the face of the main shock. Therefore this shock wave is no more a straight line in the (x, y) plane, and we are interested in the exact shock location. Taking as nondimensionalized energy $\theta = E\,C_p/R\,u_f^2$, where u_f is the longitudinal speed of the shock, and searching the disturbances of the main shock in the wave length scale $x \sim \delta\sqrt{\theta}$, for a time scale $t \sim \delta\,\theta/u_f$, the shock position is defined as

$$x_{\text{shock}} = x_{ZND} + \delta\, h\left(\frac{x}{\delta\sqrt{\theta}}\,,\,\frac{t\,u_f}{\delta\,\theta}\right).$$

Writing the Rankine–Hugoniot relations, developing all the variables in the high energy asymptotics, and supposing the wall perfectly reflecting, Buckmaster [5], Buckmaster and Ludford [7], and Buckmaster, Dold, and Schmidt-Lainé [6] derived the following evolution equation for the variable $u = 1 + h/K$ where K is a positive constant:

$$\text{(1.4)} \qquad \begin{cases} u_t + \dfrac{1}{2}(u_x)^2 = \ln\left(\dfrac{e^{cuu_{xx}} - 1}{c\,uu_{xx}}\right) + \ln u\,, \quad x \in (0, l)\,,\, t > 0, \\ u_x(0,t) = u_x(l,t) = 0\,, \quad t > 0, \\ u(x,0) = u_0(x) > 0\,, \quad x \in (0, l), \end{cases}$$

where c is a nondimensionalized positive constant representing the chemical properties (it cannot be eliminated) and l is a nondimensionalized positive constant representing the geometrical properties. Typically, for a detonation whose overdrive coefficient is $D = 1.2$, a perfect gas of massic heat ratio $\gamma = 1.2$ and a nondimensionalized heat of reaction $\overline{Q} = Q/RT_f = 50$, we have $c = 0.268$. Due to the change of unknown from h and u, the nonperturbed ZND wave is represented by the stationary constant solution $u_0 \equiv 1$.

In [6], Buckmaster, Dold, and Schmidt-Lainé briefly studied the quenching phenomenon: numerical calculations show that the term uu_{xx} in (1.4) is small and thus they neglect the terms

$$-\frac{1}{2}(u_x)^2 + \ln\left(\frac{e^{cuu_{xx}} - 1}{c\,uu_{xx}}\right)$$

in (1.4) and perform what they call a final time analysis. Then they show that the singular behavior of the perturbation shock speed, namely u_t, creates a perturbation of the postshock pressure which is also unbounded at the point $x = 0$. Thus for time close to the quenching time and for small x, the postshock pressure increases significantly above the steady state value. They then suggest that this excess pressure will be relieved by the generation of transverse shock waves, thus initiating the triple point characteristics of unstable detonation waves.

In the present work we study the asymptotic behavior of the solution u near the quenching time without neglecting the term uu_{xx}, which is shown to be small in the course of the analysis, thus giving a solid foundation to the singular perturbation approximation of [6]. This entails the use of a varied range of analytical tools now available for the study of asymptotic phenomena as well as the development of new tools. A natural extension of (1.4) is obtained by considering $f : \mathbb{R} \to \mathbb{R}$ to be a real function belonging to $\mathbf{C}^{\infty}(\mathbb{R})$ and satisfying $f(0) = 0$, $f' > 0$, and by studying the model problem (1.1). We recover (1.4) by taking $f(s)$ from (1.2). Next, a precise analysis allows us to describe the final profile of the detonation front. Finally, we prove that the singularity is essential in this model and the solution cannot be extended beyond the quenching time even in a generalized sense.

Let us mention that another aspect of (1.1) has been previously studied by one of the authors and his coworkers. The paper [4] discusses the stability of the unique constant stationary solution $u_0 \equiv 1$. First, a global bifurcation phenomenon of stationary solutions is exhibited. Then, using the geometric theory for fully nonlinear parabolic equations of Da Prato and Lunardi [28], the authors show that the constant stationary solution is unstable and they study its local invariant manifolds.

2. Outline of results. The phenomenon of quenching which occurs when the solution touches down the level $u = 0$ is due to the influence of the lower-order term $\ln u$ in (1.1), which is not bounded at the singular level $u = 0$. Moreover, at $t = T$ the equation ceases to be uniformly parabolic. The time T at which such an effect occurs is called the quenching time and the points x where it occurs are referred to as *quenching points*.

Quenching and *blow-up* are two typical kinds of singularity formation which may occur in nonlinear parabolic equations. Quenching has the peculiar feature that the solution stays bounded while some of the derivatives blow up. An example of this type of singularity was proposed by Kawarada in [23]. He considered the following semilinear parabolic equation:

$$u_t = u_{xx} + (1-u)^{-1}, \quad x \in (-a, a)\,, \, t > 0, \tag{2.1}$$

with $u(-a, t) = u(a, t) = 0$, and $u(x, 0) \equiv 0$. If the interval length a is sufficiently large, the solution u quenches, i.e., reaches the singular level 1 in a finite time. Such a singularity has been studied for semilinear and quasi-linear heat equations in several space dimensions; see [26] for a survey. In general, the quenching problem admits a standard blow-up formulation for a different semilinear parabolic equation derived by setting $(1-u)^{-1} = v \to \infty$ as $u \to 1^-$, so that $v(x, t)$ blows up in the L^∞-norm.

In this framework, (1.1) presents two mathematical novelties: first, it is a fully nonlinear equation, since the highest space derivative u_{xx} is contained in the argument of the nonlinearity f. The parabolicity of the equation comes from the fact that $f' > 0$. Nevertheless, the equation is also degenerate parabolic, because as $uu_{xx} \to 0$ there holds $f(cuu_{xx}) \sim \frac{c}{2} uu_{xx}$, and as $u \to 0$ (if u_{xx} stays bounded)

$$\frac{\partial f(cuu_{xx})}{\partial(u_{xx})} = f'(cuu_{xx})\, cu \to 0,$$

so that the equation is not uniformly parabolic near $\{u = 0\}$. Therefore, previous results do not apply, even for the proof of finite-time quenching, and we are forced to look for other ingredients to study the behavior of the solution.

On the other hand, the asymptotic finite-time behavior near the singularity, as $t \to T^-$, for quasi-linear parabolic equations is a difficult problem and possesses some common features in all the cases of blow-up, extinction, or quenching. One of the most important peculiarities of such singularities is that the limit $t \to T^-$ is often described by a singular perturbed nonlinear first-order equation, so that special techniques from singular perturbation theory are necessary.

Another question of no less interest (in particular, for physical applications) is how to extend the solution beyond the singularity, for $t > T$. It turns out that for the one-dimensional parabolic equations, this question is solved in the maximal generality for all the types of singularities. See in that respect [17], [18] where necessary and sufficient conditions for a nontrivial extension beyond singularity are established.

In the present paper, we will address the following questions for problem (1.1), (1.2):

(i) occurrence or nonoccurrence of quenching;

(ii) the asymptotic behavior of the solution as $t \to T^-$, when it quenches (of course, this imposes certain restrictions in the initial data) and of the final-time profile $u(x, T^-) > 0$ for small $x > 0$; and

(iii) existence or nonexistence of a nontrivial extension of a solution beyond quenching, for $t > T$. In other words, we want to know whether the natural extension of the solution after quenching is identically zero or not. In the former case, where $u(x, T^+) \equiv 0$, we say that there is a complete singularity.

Here is a detailed outline of the contents of the paper. Section 3 covers the local existence and regularity results for (1.1). We then address the first basic question about the behavior of the solutions, i.e., finding sufficient conditions for the occurrence or nonoccurrence of quenching. Section 4 answers this question by means of comparison techniques with lower and upper solutions. We thus prove that quenching occurs for a wide class of initial data, i.e., singular quenching is generic in this detonation model.

In section 5 we deal with the first asymptotic result, the type of quenching: we prove that single-point quenching occurs in this model. As the second asymptotic problem, supposing that $u(x, t)$ quenches at time T, we want to describe the behavior of u as $t \to T^-$. In fact, we are in the presence of a very special type of singularity because as $t \to T$ we have $\min_x u(x, t) \to 0$, so that not only u_t ceases to exist but also (1.1) is no longer uniformly parabolic. Therefore, in order to study the behavior of u near the quenching time, we will first derive a semiconvexity estimate on u_{xx} inspired by Aronson–Bénilan's estimate for the porous media equation [2] and a simultaneous estimate on u. But, unlike the classical porous medium case, now two estimates are coupled by a singular two-dimensional dynamical system, and this implies a delicate mathematical argument that we develop in section 6.

We may now proceed with the asymptotic behavior of the solution near the quenching time. We impose some conditions on the initial data which guarantee that the solution quenches in finite time. In a first step the above estimates are used to rescale the solution u and the variables x, t and thus obtain a regular equation in the neighbourhood of T. Rescaling is a quite popular and powerful technique in mechanics; see many examples in Barenblatt's book [3] and in [31] devoted to blow-up behavior in quasi-linear reaction-diffusion equations. It is worth mentioning that in the case of this fully nonlinear equation no exact self-similar solution exists, and moreover the asymptotic quenching behavior is proved to be approximately self-similar, since the second-order diffusion-like term disappears in the limit and provides only an eventual parabolic regularity of the evolution orbits. Full details are given in section 7. In order to study the behavior of the solution of the rescaled equation, we will use a technique of two of the authors [14], [16] based on the comparison of the ω-limit sets of an infinite-dimensional dynamical system and its asymptotically small singular perturbations, which allows us to study the long time behavior of a number of parabolic equations. Finally, we show that quenching occurring as $t \to T$ at the origin $x = 0$ forms the following limit profile:

$$u(x, T^-) \sim x^2 \, |\ln x| \quad \text{for small} \quad x > 0.$$

In section 8 we present numerical results to show that the current study is valid for the particular problem of a detonation in ducts.

Finally, in section 9 we prove that the solutions with the quenching singularity at $t = T$ admit no extension beyond it, for $t > T$. Using the general results from [17], [18] on extended semigroup theory, we establish that in the generic situation this quenching singularity is *complete*, in the sense that the uniquely defined *proper* solution (also called *maximal* solution) takes everywhere the singular value $u \equiv 0$ for $t > T$. Therefore, any nonnegative extension is entirely singular for $t > T$. The completeness of the singularity at $t = T$ means that the present model does not apply for $t > T$, and, in general, should be replaced.

3. Local existence and regularity. The present equation falls into the scope of the so-called *fully nonlinear parabolic equations*, which can be written in the general form

$$u_t = F(u, u_x, u_{xx}), \tag{3.1}$$

where $F(u, p, q)$ is a C^k smooth function, $k \geq 1$, defined in an open set of $\mathbf{R}^3$, and F obeys a parabolicity condition with respect to the last argument q, $F_q(u, p, q) > 0$. Several approaches for local solvability of the problem are available in the literature. In her book [27, sect. 8.5], Lunardi discusses local classical solvability of this problem using the theory of analytic semigroups. This approach has its origins in the work of Da Prato and Grisvard [29], who introduced the use of maximal regularity properties of analytic semigroups in interpolation spaces; in the case of fully nonlinear equations the construction of solutions is performed by studying the linearized equation around the initial data u_0 and then using a fixed point theorem. See also Angenent [1]. In our case F is C^∞-smooth in its domain $\{u > 0\}$ and is parabolic uniformly in sets of the form $\{0 < c \leq u \leq C\}$. This makes it possible to apply the above theory. Let us summarize the results that are of interest here.

The local existence of the solution to (1.1) is proved as in Theorem 7.4 in [29]. The following result holds.

THEOREM 3.1. *Let $u_0 \in h^{2+\theta}([0, l])$ for some $\theta \in (0, 1)$ be such that $u_0'(0) = u_0'(l) = 0$, and $u_0(x) > 0$ in $[0, l]$. Then there exist a time $T_0 > 0$ and a unique $u \in \mathbf{C}([0, T_0]; h^{2+\theta}([0, l])) \cap \mathbf{C}^1([0, T_0]; h^\theta([0, l]))$, solution of (1.1) in the classical sense in the domain $Q = (0, l) \times (0, T_0)$. Moreover, $u > 0$ in $\overline{Q} = [0, l] \times [0, T_0]$.*

We recall that for $\theta \in (0, 1)$ and n an integer we define the little Hölder spaces (cf. [27, sect. 0.2]) as

$$h^{n+\theta}([0, l]) = \left\{ u \in \mathbf{C}^n([0, l]) \,,\, \lim_{t \to 0} \max_{x, y \in [0,1], |x-y| \leq t} \frac{|u^{(n)}(x) - u^{(n)}(y)|}{|x - y|^\theta} = 0 \right\}.$$

These spaces are Banach spaces endowed with their natural norm

$$u \in h^{n+\theta}([0, l]) \,,\; \|u\| = \sum_{i=0}^{n} \max_{x \in [0,l]} |u^{(i)}(y)| + \max_{t>0,\, |x-y| \leq t,\, x, y \in [0,l]} t^{-\theta} |u^{(n)}(x) - u^{(n)}(y)| \,.$$

The little Hölder spaces and subspaces thereof appear as interpolation spaces for the Laplacian with Neumann boundary conditions in one dimension; cf. [27, sect. 3.1].

We remark that the Neumann boundary conditions allow us to extend the solution defined in spatial domain $0 \le x \le l$, first by symmetry to $-l \le x \le l$, and then by periodicity to all of $\mathbb{R}$. Different approaches to prove classical solvability of (1.1) can be found in the books [24] and [8]; see also references therein.

Since $f \in \mathbf{C}^\infty(\mathbb{R})$, one can formally differentiate (1.1) with respect to x and t and obtain equations of the quasi-linear type to which we can apply the improved existence and regularity results which hold for this type of equation; cf. [1], [27]. We thus obtain the following regularity result.

THEOREM 3.2. *With the same assumptions as in Theorem* 3.1*, for any* $0 < \epsilon < T_0/2$*, the solution* u *of* (1.1) *belongs to* $\mathbf{C}^\infty([0,l] \times [\epsilon, T_0 - \epsilon])$.

According to this theorem, a solution u of (1.1) exists globally in time, i.e., $T_0 = \infty$, if it does not blow up in finite time and does not quench.

4. To quench or not to quench. It is easy to see that the solution u of (1.1) does not blow up in finite time, i.e., $u(x,t)$ stays uniformly bounded on $(0, T_0)$. Indeed, the lower-order term, $\ln(u)$, is a sublinear function for $u \gg 1$, hence the absence of blow-up is shown by comparison with an upper solution which does not depend on x. Thus, if the solution is not global it must quench. We will find sufficient conditions on the initial data u_0 and the function f that ensure that the solution quenches, or that it does not quench, by using comparisons with suitable upper and lower solutions. We will show that these conditions are verified in the case of the detonation in ducts.

Once we prove local in time existence of classical smooth solution, we can use the maximum principle as for the linear parabolic equations; see, e.g., [10]. It will be convenient to use the notation

$$\mathbf{L}(v) = -v_t - \frac{1}{2}(v_x)^2 + f(c\, v\, v_{xx}) + \ln v$$

for sufficiently smooth functions $v : [0,l] \times \mathbb{R}^+ \to \mathbb{R}^+$.

4.1. Flat upper and lower solutions. If we compare the solution u of (1.1) with flat profiles we obtain the following result.

PROPOSITION 4.1. *Let* $u_0 \in h_\infty^{2+\theta}([0,l])$ *for some* $\theta \in (0,1)$ *satisfy* $u_0'(0) = u_0'(l) = 0$ *and* $u_0(x) > 0$ *in* $[0,1]$. *Let* $m = \min_x u_0(x)$, $M = \max_x u_0(x)$. *We consider* u^- *and* u^+*, the solutions of the ODEs*

$$(u^-)' = \ln(u^-), \quad t > 0, \quad u^-(0) = m, \tag{4.1}$$

$$(u^+)' = \ln(u^+), \quad t > 0, \quad u^+(0) = M, \tag{4.2}$$

and we denote by T^- *and* T^+ *their respective maximal time of existence. Let* $T = \min\{T^-, T^+, T_0\}$*. Then* $u^-(t) \le u(x,t) \le u^+(t)$ in $[0,l] \times [0,T]$.

Proof. This result is a straightforward consequence of the maximum principle for parabolic equations since we know that $\mathbf{L}(u^+) = \mathbf{L}(u^-) = 0$ and $u^-(0) \le u(x,0) \le u^+(0)$ in $[0,l]$. ◻

COROLLARY 4.1. *With the same assumptions as in Proposition* 4.1*, we have the following:*

1. *The solution* u *of* (1.1) *never blows up in finite time;*

2. *if* $m > 1$*, the solution* u *does not quench; therefore, it is a global solution, i.e.,* $T_0 = \infty$*;*

3. *if $M<1$, the solution u quenches and the quenching time T satisfies $T_m \le T \le T_M$ with*

$$T_m = \int_{-\ln\, m}^{\infty} e^{-z}\, dz/z\ ,\ T_M = \int_{-\ln\, M}^{\infty} e^{-z}\, dz/z\,.$$

The result follows by integrating the ODEs (4.1) and (4.2).

In the case where u_0 oscillates around 1, u can quench or not, depending on the initial data u_0 and the nonlinearity f, as discussed in the next subsection. Since the problem is parabolic, we will use comparison with suitable parabolic upper and lower solutions to prove existence or nonexistence of quenching. First, since (1.1) is invariant under the reflection with respect to the origin, we consider symmetric initial data, $u_0(-x) \equiv u_0(x)$, satisfying $u_0 \in h_\infty^{2+\theta}([0,l])$ with $u_0'(0) = u_0'(l) = 0$, and $u_0(x) > 0$ for $x \in [0,l]$. By the symmetry and the uniqueness we then obtain that the solution $u(x,t)$ is even in x for $t>0$. Therefore, it can be considered as the solution of (1.1) in $Q = (0,l) \times \mathbb{R}^+$ with the Neumann (symmetry) boundary condition at the origin

$$u_x(0,t) = 0, \quad t > 0. \tag{4.3}$$

4.2. No quenching when u_0 crosses 1. We will assume in this section that *f is convex over* $\mathbb{R}$ and we will find in Q a stationary lower solution $\underline{u}$ independent of t defined by

$$\underline{u}(x) = a - b\cos(\lambda x), \quad \lambda = \frac{\pi}{l},$$

with the coefficients $a > b > 0$ satisfying

$$0 < a-b < 1, \quad a+b > 1, \tag{4.4}$$

so that $\underline{u}(x)$ intersects the stationary solution $u \equiv 1$. The positive lower solution satisfies the Neumann boundary conditions at $x=0$ and $x=l$. If we can find a lower solution like that then $\underline{u} \le u$ in Q implies that u does not quench. We now assume that

$$f(s) \ge s/2 \quad \text{in} \quad \mathbb{R},$$

the inequality satisfied by (1.2). We then conclude that the inequality $\mathbf{L}(\underline{u}) \ge 0$ in $[-l,l]$ holds if

$$\frac{1}{2}(\underline{u}_x)^2 \le \frac{1}{2}c\underline{u}\,\underline{u}_{xx} + \ln \underline{u} \quad \text{in} \quad [0,l].$$

Substituting the lower solution, we arrive at the inequality

$$-\frac{1}{2}b^2\lambda^2(1-z^2) + \frac{1}{2}bc\lambda^2(a-bz)z + \ln(a-bz) \ge 0$$

for all $z = \cos(\lambda x) \in [-1,1]$. Using the estimate

$$\ln(a-bz) \ge \ln(a-b) + b(1-z)d, \quad z \in [-1,1]; \ \ d = \frac{1}{2b}[\ln(a+b) - \ln(a-b)] > 0,$$

we finally arrive at the inequality

$$\Phi(z) \equiv Az^2 + Bz + C \le 0, \quad z \in [-1,1],$$

where $A = b^2\lambda^2(1+c)/2$, $B = bd - abc\lambda^2/2$, and $C = b^2/2 - \lambda^2 \ln(a-b) - bd$. Since Φ is convex, we then deduce the following conditions on the coefficients:

$$\Phi(\pm 1) = A \pm B + C \leq 0. \tag{4.5}$$

PROPOSITION 4.2. *Let $\theta \in (0,1)$, $u_0 \in h_\infty^{2+\theta}([0,l])$ satisfying $u_0'(0) = u_0'(l) = 0$, and $u_0(x) > 0$ in $[0,l]$. Let the function f be convex from $\mathbb{R}$ onto $\mathbb{R}$ and $f(s) \geq s/2$ in $\mathbb{R}$. Let (a,b) be a solution of inequalities* (4.4), (4.5). *If $u_0(x) \geq \underline{u}(x)$ in $[0,l]$, then $u(x,t) \geq \underline{u}(x)$ in Q, and the solution u of* (1.1) *does not quench, and it is a global classical solution.*

Let us study the solvability of the above system (4.4), (4.5) of algebraic inequalities. We consider the case $a \approx 1^+$ and $b = \alpha a \approx 0$ where $\alpha > 0$ is a small constant. Then inequalities (4.5) reduce to

$$a^2\lambda^2\alpha^2[\alpha(1+c/2)-c/2] \leq \ln a + \ln(1-\alpha), \quad a^2\lambda^2\alpha^2[\alpha(1+c/2)+c/2] \leq \ln a + \ln(1+\alpha).$$

Setting now $\alpha = 1+\epsilon$, $0 < \epsilon \ll 1$, and assuming that $0 < \alpha \ll 1$, in the first approximation we get $\lambda^2\alpha(-c/2) < \epsilon - \alpha$, $\lambda^2\alpha(c/2) < \epsilon + \alpha$, from which we get the following bound for $\eta = \epsilon/\alpha$: $\max\{1 - \lambda^2 c/2,\ \lambda^2 c/2 - 1\} < \eta < 1$. These are valid provided that $c/\lambda^2 < 4/\pi^2$, which is a particular sufficient condition of the solvability.

4.3. Quenching when u_0 crosses 1. Throughout this section we suppose that the function f satisfies

$$f(s)/s \quad \text{is bounded} \quad \text{and} \quad f(s) \leq s \quad \text{for} \quad s > 0.$$

Both assumptions are true for the particular choice (1.2) since $f'(0) = 1/2$, $f'(\infty) = 1$, and f is convex. We suppose that $c \in (0,1)$; cf. section 2.

We will find an upper solution $\overline{u}$ defined in Q by

$$\overline{u}(x,t) = a_1(t) + a_2(t)\,x^2,$$

where the functions a_1 and a_2 satisfy these conditions: there exists $T > 0$ such that

$$a_1(t) > 0\,,\ a_2(t) > 0 \ \text{ for } t \in [0,T), \quad \text{and} \ \lim_{t\to T} a_1(t) = 0. \tag{4.6}$$

Observe that $\overline{u}_x(0,t) \equiv 0$ and $\overline{u}_x(l,t) = 2a_2 l > 0$, so that the necessary inequalities on the boundary are valid. If we can find two functions a_1, a_2 such that $\overline{u}(x,t) \geq u(x,t)$ in $Q_T = Q \cap \{t < T\}$ and

$$\min_{x\in[-l,l]} \overline{u}(x,t) = a_1(t) \to 0 \text{ as } t \to T^-,$$

then $u(x,t)$ quenches at a time $T_0 \leq T$.

Using the maximum principle for parabolic equations, we will find two functions a_1, a_2 such that $\mathbf{L}(\overline{u}) \leq 0$ in Q_T, and we will suppose that $u_0(x) = u(x,0) \leq a_{10} + a_{20}x^2$ in $[0,l]$ where $a_{i0} = a_i(0)$. Recalling that we are in the case where u_0 crosses 1, i.e., $m = \min_x u_0(x) < 1$ and $M = \max_x u_0(x) > 1$, we first have

$$0 < a_{10} < 1, \qquad 1 < a_{10} + a_{20}\,l^2. \tag{4.7}$$

With the assumption on f and using the obvious inequality $\ln(y) \leq y - 1$ for $y > 0$, we obtain that in Q_T

$$\mathbf{L}(\overline{u}) \leq -\overline{u}_t - \frac{1}{2}(\overline{u}_x)^2 + c\,\overline{u}\,\overline{u}_{xx} + \overline{u} - 1.$$

Therefore, it is sufficient to find a_1 , a_2 such that in Q

$$-\overline{u}_t - \frac{1}{2}(\overline{u}_x)^2 + c\,\overline{u}\,\overline{u}_{xx} + \overline{u} - 1 \le 0.$$

Taking the equality in the above inequalities, after some algebraic manipulations we finally arrive at the following two-dimensional dynamical system:

$$a_1' = 2c\,a_1\,a_2 + a_1 - 1, \quad a_2' = a_2 - 2\,(1-c)\,a_2^2. \tag{4.8}$$

The second equation is solved separately to give

$$a_2(t) = \frac{e^t}{A + 2(1-c)e^t}, \quad A \in \mathbb{R}.$$

Then from the first one we derive that

$$a_1(t) = (A + 2(1-c)e^t)^{c/(1-c)} \left[B - \int_0^t e^{-\xi}(A + 2(1-c)e^{\xi})^{c/(1-c)} d\xi \right],$$

where B is a constant. Fix an $A > 0$. Since $c \in (0,1)$, (4.6) holds provided that

$$B < \int_0^{\infty} e^{-\xi}(A + 2(1-c)e^{\xi})^{c/(1-c)} d\xi \equiv F(A). \tag{4.9}$$

Since the coefficients a_{10}, a_{20} verify $a_{20} = (A + 2(1-c))^{-1}$, $a_{10} = (A + 2(1-c))^{c/(1-c)}B$, the inequality (4.9) means that

$$a_{10}a_{20}^{-c/(1-c)} < F(a_{20}^{-1} - 2(1-c)), \tag{4.10}$$

i.e., the initial point (a_{10}, a_{20}) lies below the *separatrix* on the phase portrait of the system (4.8).

We now summarize our results.

PROPOSITION 4.3. *Suppose that $f(s) \le s$ in $\mathbb{R}^+$ and $c \in (0,1)$. Let a_{10} , a_{20} satisfy inequalities* (4.7) *and* (4.10). *Let $a_1(t)$, $a_2(t)$ be the solution of the dynamical system* (4.8) *with initial data a_{10} , a_{20}. Let $\overline{T} = \min\{T > 0 \,|\, a_1(T) = 0\}$. If $u_0(x) \le a_{10} + a_{20}\,x^2$ in $[-l,l]$, then $u(x,t) \le a_1(t) + a_2(t)\,x^2$ in $[-l,l] \times [0,\overline{T})$, and u quenches in a time $T \le \overline{T}$.*

This result is valid for the problem of detonation in ducts.

5. Single point quenching and first sharp estimate. We now prove that the singular (unbounded) absorption term of (1.1), (1.2), $\ln u$, is strong enough to produce *single point quenching* in the presence of the fully nonlinear diffusion term. This means that the *quenching set*

$$E(u_0) = \{x \in [0,l] \,|\, \exists \, \{x_n\} \to x, \ \{t_n\} \to T^- \text{ such that } \ u(x_n, t_n) > 0\}$$

consists of a single point. Therefore, the other singular phenomena like *regional* or *global* quenching as $t \to T$ are not possible. Several techniques to solve this asymptotic problem are known in the literature. In studies of blow-up, it is called the *localization* problem; see [31, Chap. 4], [18], [19], and references therein.

We consider a function $u(x,t)$ which is a classical positive solution of (1.1) in $Q_T = (0,l) \times (0,T)$ and quenches at a finite time $T > 0$:

$$\min_{x \in (0,l)} u(x,t) \to 0 \quad \text{as} \ \ t \to T. \tag{5.1}$$

Next we use the following eventual monotonicity result which is true for any finite-time singularity like blow-up, extinction or quenching. See the proof in [15, sect. 10] (extinction) or in [19, sect. 2] (blow-up). Denote by $U_T(t) > 0$ on $[0, T)$ the unique solution of the ODE

$$U_T' = \ln U_T, \quad t \in (0, T); \quad U_T(T) = 0.$$

Then $u(x, t)$ must intersect $U_T(t)$ for all $t \in [0, T)$ (otherwise they cannot have the same finite quenching time T), and moreover we have the next proposition.

PROPOSITION 5.1. *Let $u_0(x)$ intersect $U_T(0)$ exactly once. Then $u(x, t)$ is strictly monotone increasing in x near the quenching point $x \approx 0$, $t \approx T^-$.*

Therefore, for $t \approx T^-$,

$$u_x > 0 \quad \text{for all } x > 0 \text{ small,} \tag{5.2}$$

so that without loss of generality we may assume that the smooth initial data u_0 are strictly monotone,

$$u_0' > 0, \quad x \in (0, l). \tag{5.3}$$

Then (5.2) holds in Q_T. We now prove that the solution quenches at a single point. By the monotonicity, it is indeed the origin $x = 0$.

PROPOSITION 5.2. *Under the above assumptions*

$$E(u_0) = \{0\}. \tag{5.4}$$

Proof. Differentiating (1.1) in x, we obtain that $v = u_x$ solves in Q_T the quasilinear equation

$$v_t = -vv_x + cf'(cuv_x)(uv_{xx} + vv_x) + \frac{v}{u}. \tag{5.5}$$

Since $v > 0$ by (5.2), and $f' > 0$, we obtain the parabolic differential inequality

$$v_t \geq cf'(cuv_x)(uv_{xx} + vv_x) - vv_x.$$

In a neighborhood of the quenching point $x = 0$, $t = T^-$ we have $0 < u \ll 1$, $0 < v \ll 1$. Using a standard comparison, we also may suppose that $v_x \geq 0$. Using the estimates on the function (1.2)

$$s/2 \leq f(s) \leq s, \quad s \geq 0, \tag{5.6}$$

we finally obtain the inequality

$$v_t \geq av_xv_{xx} - vv_x \quad \text{in } Q_T, \tag{5.7}$$

where $a(x, t) = \alpha c^2u^2 > 0$ and $\alpha = 1/2$ or 1 depending on $v_{xx} \geq 0$ or $v_{xx} \leq 0$. In both cases we conclude that $v = u_x$ is a smooth supersolution (see [22]) of the gradient diffusivity equation with a quadratic convection term

$$V_t = \frac{1}{2}a(|V_x|V_x)_x - \frac{1}{2}(V^2)_x \quad \text{in } Q_T,$$

with the same positive initial data $v_0 \equiv u_0'$. That the solution V is strictly positive is proved by comparison with the exact separable solution $V_*(x, t) = x/(t + C_0)$, where the constant $C_0 \gg 1$ is chosen so that $V_*(x,0) \geq v_0(x)$ (this estimate is true for the regular profile $u_x(x,\tau)$ with an arbitrarily small time-shifting parameter $\tau > 0$). Then $V \geq V_*$ in Q_T by the standard comparison for degenerate parabolic equations [22], and finally, $v \geq V > 0$ in $\overline{Q}_T$ by the maximum principle. Hence

$$v(x,T^-) \equiv u_x(x,T^-) \geq x/(T+C_0) > 0 \quad \text{for all } x > 0.$$

Integrating this inequality yields

$$u(x,T^-) \geq x^2/2(T+C_0) > 0 \quad \text{for } x > 0, \tag{5.8}$$

whence single point quenching at $x = 0$ only. ∎

Using Proposition 5.2, we now derive an important optimal bound on the behavior of the solution near the singular quenching point. We apply a gradient estimate method first proposed in [11] in the study of blow-up solutions to semilinear heat equations. This method admits a natural generalization to quasi-linear equations; see [13], [19], and the references therein. We present a brief description of this technique. We mainly follow the construction given in section 4 in [19]. Since we study the quenching phenomenon of the solutions which become arbitrarily small as $t \to T$ in a neighborhood of the origin, without loss of generality we now may assume that $0 < u_0 \leq 1/2$, and hence $0 < u \leq 1/2$ in Q_T by the maximum principle. Given a quenching solution u, we consider the function

$$J = u_x - xF(u),$$

where $F(u) > 0$ for $u > 0$ is a smooth function to be determined later. Using the maximum principle, we derive conditions under which the function J is nonnegative in a neighborhood of the quenching point: $J \geq 0$ in $[0,\delta] \times [T-\delta, T)$. One can derive from (1.1) a linear parabolic equation satisfied by J. Namely, we have

$$J_t = u_{xt} - xF'(u)u_t. \tag{5.9}$$

The second-order derivative u_{xt} is calculated from (1.1):

$$u_{xt} = cf'(cuu_{xx})(uu_{xxx} + u_x u_{xx}) - u_x u_{xx} + \frac{u_x}{u}.$$

Evaluating the derivatives $u_x = J + xF$, $u_{xx} = J_x + xF'J + F + x^2FF'$, and

$$u_{xxx} = J_{xx} + xF'J_x + (2F' + xF''u_x + x^2(FF')')J + 3xFF' + x^3F(FF')',$$

we substitute them into (5.9). We then obtain a fully nonlinear equation for a sufficiently smooth solution J. Using Lagrange's formula of finite increments, one can write it in a quasi-linear form (the so-called formal linearization procedure)

$$J_t = \mathbf{L}J + q, \tag{5.10}$$

where $\mathbf{L}$ is a second-order (elliptic) ordinary differential operator with nonconstant smooth coefficients, and q is the nonhomogeneous term independent of J. It follows from the above linearization procedure that it takes the form (see [13] and [19, sect. 4])

$$q = \{u_{xt} - xF'(u)u_t\}_{J \equiv 0}.$$

This means that in calculating q, we may substitute the derivatives in the form

$$u_x = xF, \quad u_{xx} = F + x^2 FF', \quad u_{xxx} = 3xFF' + x^3 F(FF')'.$$

By the maximum principle, it follows from (5.10) that the inequality $q \geq 0$ is the required condition, which together with the condition $J \geq 0$ on the parabolic boundary guarantee $J \geq 0$. The function q consists of the two terms:

$$q = x\, I_1 + x^3\, I_2,$$

where

$$I_1 = cf'(z)(3uFF' + F^2) - F^2 - F' f(z) + \frac{F}{u} - F' \ln u \tag{5.11}$$

and

$$I_2 = cf'(z)(uF(FF')' + F^2 F') - \frac{1}{2} F^2 F'. \tag{5.12}$$

The argument z in the fully nonlinear terms is given by $z = cuF + cx^2 uFF'$. Thus, $q \geq 0$ if $I_1 \geq 0$, $I_2 \geq 0$ for $x \in [0, \delta]$, and $u \in (0, 1/2]$.

As in the case of quasi-linear equations [13], [19], the main idea of the method consists in solving the first inequality in a neighborhood of the singular quenching point:

$$I_1 \geq 0 \quad \text{for small} \quad u > 0 \;\text{ and }\; x > 0. \tag{5.13}$$

We establish that this inequality has a suitable solution and, moreover, determines an *optimal* choice of the unknown function $F(u)$. Let us first define a function F_0 from the ODE containing the last two terms in (5.11), which will be shown to be the leading ones as $u \to 0$:

$$F_0' \ln u = \frac{F_0}{u} \quad \text{for small} \quad u > 0.$$

This yields $F_0(u) = \epsilon(-\ln u) > 0$, $\epsilon > 0$. Then $F_0' = -\epsilon/u < 0$, $F_0'' = \epsilon/u^2 > 0$, so that F_0 is convex for small $u > 0$. This property plays an important role. Let us show that (5.13), (5.11) admit a solution F being a small perturbation of the function F_0. We now are looking for a solution of (5.13) in the form $F(u) = F_0(u)(1 + o(1))$ for small $u > 0$, with similar expressions for the first two derivatives of F. In order to estimate the fully nonlinear terms, we use known properties of the function $f(z)$ such as

$$f(z) \leq f(cuF) \leq cuF = 2c\epsilon u(-\ln u), \quad f(z) \geq f(cx^2 uFF') \geq \frac{1}{2} cx^2 uFF', \quad f'(z) \leq 1,$$

etc. Using such inequalities in (5.11) we obtain the following inequality for the function F: uniformly for $u \in [0, \delta]$:

$$I_1 \geq \frac{F}{u} - F' \ln u + O(\ln^2 u) \geq 0.$$

Therefore, solving this with the equality sign, we prove that there exists the following optimal solution:

$$F(u) = F_0(u) + O(u \ln u) = F_0(u)(1 + O(u)). \tag{5.14}$$

The second inequality, $I_2 \geq 0$, is valid due to the monotonicity and convexity of F and the condition $f' > 0$. This implies that such a solution F exists.

It follows from Proposition 5.2 that $u(\delta, t) > 0$ for all $t \in [T - \delta, T]$ so that the solution is sufficiently smooth near that positivity point. Hence $u_x(\delta, t) \geq c_1 > 0$ there. Therefore, using (5.14), we deduce that $J gec_1 - 2\delta\epsilon(-\ln u) \geq 0$ at $x = \delta$, $t \in [T - \delta, T]$, provided that $\epsilon > 0$ is sufficiently small. Similarly, at $t = T - \delta$ there holds $J \geq u_x - 2\epsilon x(-\ln u) \geq 0$ on $[0, \delta]$ for ϵ small. Finally, we arrive at the following result: there exists a small $\epsilon > 0$ such that

$$J = u_x - \epsilon x(-\ln u) \geq 0 \quad \text{in} \quad [0, \delta] \times [T - \delta, T). \tag{5.15}$$

Integrating (5.15) over $(0, x)$, we obtain

$$\frac{z}{(-\ln z)}\Big|_{u(0,t)}^{u(x,t)} \geq \int_{u(0,t)}^{u(x,t)} \frac{dz}{(-\ln z)} \geq \frac{1}{2}\epsilon x^2,$$

whence the following estimate from below of the spatial profile of the solution $u(x, t)$ near singularity:

$$\frac{u(x,t)}{(-\ln u(x,t))} \geq \frac{u(0,t)}{(-\ln u(0,t))} + \frac{1}{2}\epsilon x^2. \tag{5.16}$$

In particular, setting here $t = T^-$, so that $u(0, T^-) = 0$, we get a lower estimate of the final-time profile of the form (cf. (5.8))

$$u(x, T^-) \geq \epsilon x^2 |\ln x| > 0 \quad \text{for small} \quad x > 0. \tag{5.17}$$

It will be shown that the above estimate corresponds to the actual behavior of the final-time profile.

PROPOSITION 5.3. *Under the above hypotheses, the solution vanishes at $t = T^-$ and estimate* (5.17) *holds.*

6. Fundamental estimates. We now arrive at the core of the paper, where we consider a solution $u(x, t)$ of problem (1.1) that quenches in finite time T and derive two simultaneous estimates, namely, an estimate of the actual size of u near quenching and an upper bound on u_{xx}. These two estimates will give us the information on the profile of u close to the quenching time necessary for the asymptotic study. Such a second-order estimate is a new technique which is based on the scrutiny of the evolution equation satisfied by u_{xx} together with the original equation for u. In the present case such a study cannot be done independently of the estimate for u and we arrive at a *system of differential inequalities*, whose analysis is rather delicate. A particular case of such a technique was introduced by Aronson and Bénilan for the porous media equation [2], where a single inequality for u_{xx} can be studied separately (the so-called *semiconvexity estimates*). We think that the result and the technique based on systems of inequalities could be appealing to the reader interested in regularity and a priori estimates for nonlinear parabolic equations in view of further applications. Some extensions are presented in [12].

We begin our study with the upper bound for the quantity $\min_x u(x, t)$ in the neighborhood of T which follows from simple comparison considerations. We impose the same hypotheses on the initial data as in the previous section.

PROPOSITION 6.1. *Under the stated assumptions on u, we have for $t \approx T$, $t < T$,*

$$\min_{x \in (0,l)} u(x, t) \leq (T - t)|\ln(T - t)| \left[1 + O\left(\frac{1}{\ln(T - t)}\right)\right]. \tag{6.1}$$

Proof. For $t \in [0, T)$, let us denote $U(t) = \min_x u(x,t)$. By Theorem 3.2, u is a smooth function, so for any $t \in [0,T)$ there exists $x_{min} \in [0,l]$ such that $U(t) = u(x_{min}, t)$. Since $U(t) > 0$ and $u_{xx}(x_{min}, t) \geq 0$ and $f(s) \geq 0$, using the maximum principle we have that there exists a $\delta > 0$ such that for any $t \in [T-\delta, T)$ there holds $U(t) \leq u^-(t)$, where $u^-(t)$ is the solution of the ODE

$$\frac{du^-}{dt} = \ln u^-, \quad t \in (0,T), \;\; u^-(T) = 0.$$

Now, we have

$$T - t = \int_{-\ln(u^-(t))}^{\infty} e^{-z} \, \frac{dz}{z}.$$

Let us note that

$$E_1(s) = \int_s^{\infty} e^{-z} \, \frac{dz}{z}$$

is the Schlomilch function. Using its asymptotic expansion [21, p. 32],

$$E_1(s) = \frac{1}{s} e^{-s} \left(1 - \frac{1}{s} + o\left(\frac{1}{s}\right) \right),$$

one obtains

$$T - t = -\frac{u^-(t)}{\ln(u^-(t))} \left(1 + o\left(\frac{1}{\ln(u^-(t))} \right) \right).$$

Therefore,

$$u^-(t) = (t-T)\ln(T-t)(1 + o(T-t)).$$

Thus, we arrive at (6.1). □

We proceed now with the lower estimate for $u(\cdot, t)$ as well as the upper estimate for $u_{xx}(\cdot,t)$. From Theorem 3.2, u is a smooth function, thus differentiating (1.1) with respect to x, we have that $v = u_x$ solves in Q_T (5.5) with the Dirichlet boundary conditions

$$v(0,t) = v(l,t) = 0, \quad t \in (0,T); \quad v(x,0) = u_0'(x) > 0, \quad x \in (0,l).$$

By the assumptions on f and since $u(x,t) > 0$ in Q_T, this is a quasi-linear parabolic equation which is uniformly parabolic in $Q_{T-\epsilon}$ for any $\epsilon > 0$ small. Therefore, by the maximum principle $v(x,t) > 0$ in Q_T. The single minimum of $u(x,t)$ in x is always reached at the origin $x = 0$.

In our main result we need an *extra assumption* on f:

$$c\,[f'(s) + 2f''(s)s] < 1 \quad \text{in } \mathbb{R}^+. \tag{6.2}$$

This assumption is satisfied by (1.2) with a constant $c \in (0,1)$, in particular with the constant given in section 1.

THEOREM 6.1. *Assume that u quenches in a finite time T, and u_0 and f satisfy the assumptions stated before. Moreover, assume that $W_0 = \max\{u_0''(x)\}$ is small enough so that*

$$f(c\,z_0) + \ln(u_0(0)) + 1 \leq 0, \quad z_0 = u_0(0)W_0. \tag{6.3}$$

Then we have the following estimate for u: as $t \to T$

$$\min_{x\in(0,l)} u(x,t) = (T-t)|\ln(T-t)|\left[1+O\left(\frac{1}{\ln(T-t)}\right)\right], \tag{6.4}$$

as well as the estimate for u_{xx}: given $\delta > 0$ small there exists $C > 0$ such that

$$u_{xx}(x,t) \le C|\ln(T-t)| \tag{6.5}$$

in $(x,t) \in [0,l] \times [T-\delta, T)$.

Let us note that the above condition (6.3) on the initial data not only guarantees that the solution quenches in finite time, but also that the asymptotic shape (the spatial convexity) is strong enough to exhibit the stable quenching pattern. Observe that as in the blow-up problems, [31, p. 195] (see also references therein), we can expect that there exists a countable set of different quenching patterns. We thus impose a condition to exhibit the first (generic) one.

Proof. The proof is divided into three steps.

Step 1. *Differential inequalities for* $\max u_{xx}$ *and* $\min u$. The main idea is to derive differential inequalities for the two related quantities to be estimated, the minimum of $u(\cdot,t)$ at time t, which in view of the hypotheses of the data is taken at $x=0$,

$$U(t) = u(0,t), \tag{6.6}$$

and the maximum of the spatial second derivative:

$$\widetilde{w}(t) = \max_{x\in[0,l]} u_{xx}(x,t). \tag{6.7}$$

By the existence and regularity results, Theorems 3.1 and 3.2, both functions are well defined for $0<t<T$. We already have an upper bound for U that shows how it goes to 0 as $t \to T$. In order to estimate $\widetilde{w}$, we differentiate (1.1) twice and see that the function $w = u_{xx}$ is a solution of the following partial differential equation:

$$w_t = L(x,t,w,w_x,w_{xx}) \equiv c\, f'(c\,u\,w)\,u\,w_{xx} + (2c\, f'(c\,u\,w)-1)\,u_x\,w_x$$

$$+(c\,f'(c\,u\,w)-1)w^2 + c^2 f''(c\,u\,w)\,(u_x\,w + u\,w_x)^2 + \frac{w}{u} - \frac{(u_x)^2}{u^2}. \tag{6.8}$$

We construct an upper solution $W(t) > 0$ of (6.8) independent of x. Thus assuming that

$$w = u_{xx} < W \quad \text{in } \ Q_T,$$

and substituting $W(t)$ into the parabolic differential inequality corresponding to (6.8), we conclude that $W(t)$ can be chosen so that the following first-order ordinary differential inequality holds:

$$W' \ge \sup_x L(x,t,W,0,0), \quad t \in (0,T).$$

Excluding the last nonpositive term in (6.8) and replacing u by U in the positive term w/u, we have that this inequality is valid if W solves

$$W' \ge W^2 \sup_x \left[cf'(cu(x,t)W) - 1 + c^2 f''(cu(x,t)W)(u_x(x,t))^2\right] + \frac{W}{U}.$$

Let us estimate the expression in brackets. From the monotonicity assumption on u_0, we know that $u_x(x,t) > 0$ for $0 < x < l$. Therefore, $u_x(x,t)w(x,t) \equiv u_x u_{xx} \le u_x(x,t)W(t)$. Integrating this inequality over $(0,x)$ leads to

$$0 < \frac{1}{2}(u_x(x,t))^2 \le u(x,t)\,W(t) - u(0,t)\,W(t).$$

Since $W(t) > 0$ on $[0,T)$ by the assumption, we obtain in Q_T $f''(c\,u\,W)u_x^2 \le 2f''(c\,u\,W)\,u\,W$. Denoting $s = c\,u(x,t)W(t)$, the expression in brackets is estimated as

$$c\,f'(cuW) - 1 + c^2\,f''(cuW)(u_x)^2 \le cf'(s) + 2cf''(s)s - 1.$$

Assumption (6.2) implies that this quantity is negative. Hence we can fix the following ODE equation for the upper solution:

$$W' = W/U. \tag{6.9}$$

Clearly, $U(t) \equiv u(0,t)$ is smooth and, moreover, strictly decreasing (a typical property of general solutions near singularity; see p. 420 in [31]). Hence W satisfying (6.9) is also smooth on $(0,T)$.

Next, let us derive the second differential inequality for the function U. Setting $x = 0$ in (1.1), we have $U'(t) = f(c\,U(t)\,u_{xx}(0,t)) + \ln(U(t))$. Since f is increasing and $u_{xx} < W$ we arrive at the inequality

$$U'(t) < f(c\,U(t)\,W(t)) + \ln(U(t)). \tag{6.10}$$

Step 2. *A two-dimensional dynamical system.* In order to estimate the evolution of the functions $\{U(t), W(t)\}$, we first eliminate the inequality in (6.10) and consider the two-dimensional dynamical system

$$\begin{cases} W' = W/U, \\ U' = f(c\,U\,W) + \ln U, \quad t > 0. \end{cases} \tag{6.11}$$

In the autonomous form we get the equation

$$\frac{dU}{dW} = \frac{U}{W}\,[f(c\,U\,W) + \ln U]. \tag{6.12}$$

We consider the orbits of this system starting from

$$W(0) = W_0 > 0, \quad U(0) = U_0 > 0. \tag{6.13}$$

It is clear that such orbits will stay in the first quadrant and will evolve as t increases in the direction of increasing W. As for the monotonicity in U, the zero-isocline of (6.12) is given by a monotone curve γ_0 in the WU-plane with equation $f(cUW) + \ln U = 0$, i.e.,

$$W = \frac{1}{cU} f^{-1}(\ln(1/U)).$$

This curve is monotone decreasing and joins $W = 0, U = 1$ with $W = \infty, U = 0$. In the first limit $UW \to 0$, in the second to infinity. If $f(s)/s \to \lambda_1 = 1$ as $s \to \infty$, the isocline has the development

$$W \sim \frac{\ln(1/U)}{cU}, \quad \text{or} \quad U \sim \frac{\ln W}{cW} \tag{6.14}$$

as $W \to \infty$. The region Γ_0 contained between the isocline γ_0 and the axes is a region where the orbits of (6.12) are monotone decreasing curves. If a solution starting in that region crosses γ_0, then the orbit $U = U(W)$ becomes increasing, U goes to infinity, the solution of (6.11) lives for an infinite time and it does not quench. These orbits have no interest for us, so we eliminate them from our consideration.

On the other hand, there are orbits of (6.11) which lie in the region Γ_0 for all their existence time. This is easier to see if we introduce the variable $Z = UW$. The equation for Z is

$$\frac{dZ}{dt} = \frac{Z}{U}\left[f(cZ) + \ln U + 1\right]. \tag{6.15}$$

When we consider now orbits of the system (6.11) in the ZU-plane with the autonomous equation

$$\frac{dU}{dZ} = \frac{U[f(cZ) + \ln U]}{Z[f(cZ) + \ln U + 1]},$$

we discover that the geometrical situation has a certain resemblance to the previous one, with an infinite-slope isocline given by a curve γ_1:

$$f(cZ) + \ln U + 1 = 0,$$

which lies in the ZU-plane below the curve corresponding to γ_0, which now reads $f(cZ) + \ln U = 0$. The curve γ_1 is monotone and joins the point $Z = 0, U = 1/e$ to $Z = \infty$, $U = 0$. An analysis of the flow in the regions Γ_1, contained between γ_1 and the axes, and Γ_2, contained between γ_0, γ_1 and the U-axis, shows that Γ_1 is an invariant region and that there are infinitely many solutions starting at $t = -\infty$ from $(0, 1)$ and ending as $t \to \infty$ at $(0, 0)$. Let us call this family of solutions $\mathbf{F}$. They spend their whole late life in Γ_1, which in terms of the original WU-plane implies that they stay all their life span in Γ_0. We may also prove that they quench in finite time. Indeed, since any such orbit ends up in $(0, 0)$, going back to the equation for dU/dt we get

$$dt \sim dU/\ln U,$$

which gives a convergent integral as $U \to 0$. As an upper bound for this family of orbits, there must be a separatrix, which lives in Γ_2 and joins monotonically $(0, 1)$ with $(\infty, 0)$ in the ZU-plane. This curve resembles the isocline for large Z, so that in the first approximation $\ln U \sim -cZ$. It also has a finite life span, since $dt = UdW/W$ and $U(W)$ lies below the isocline γ_0 for which, in view of the expression (6.14), there is a convergent integral and $\int dt$ is finite (this is true even if there is no upper bound on $f(s)/s$ as $s \to \infty$).

We can now take the whole set of curves $\mathbf{F}$ together with their separatrix. They fill a region $\Gamma \subset \Gamma_0$ in the WU-plane. Since they have finite life-time τ and this time depends continuously on the data, we can draw the lines of equal life-times that will cover the whole region Γ. The axis $U = 0$ corresponds to the limit of zero life-time, $\tau = 0$. On the other hand, on the vertical axis, $W = 0$, $0 < U < 1/e$, we have infinite life-time. The curves $\tau(W_0, U_0) =$constant form a family of curves which start at $(W_0 = 0, U_0 = 0)$ and end somewhere at the separatrix. They cover the whole region Γ (see Figure 1). It will be useful to have some criterion on the initial location that

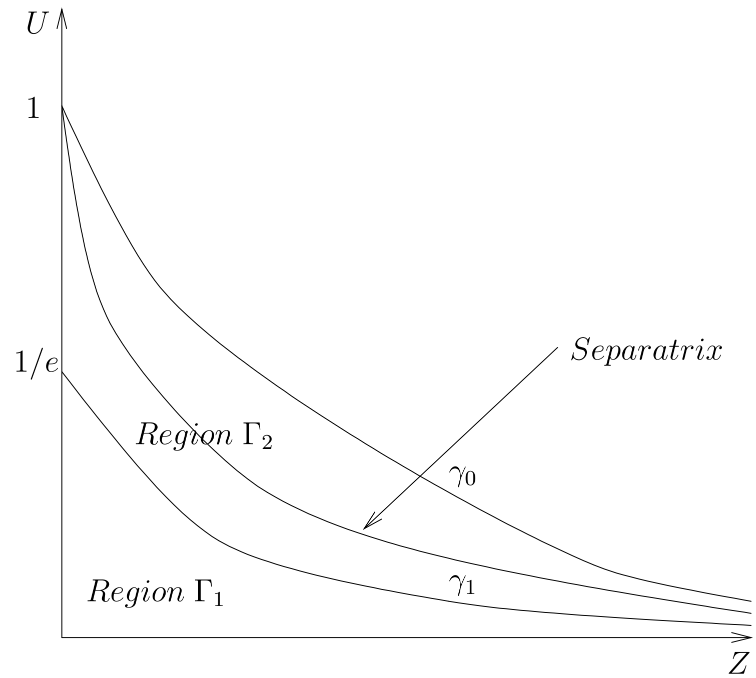

FIG. 1. *Invariant regions.*

ensures that the orbit is in $\mathbf{F}$, i.e., is such that $Z \to 0$. It is easy to see from the system that, once $Z' \le 0$ at one instant, then $Z \to 0$. This means asking that

$$f(cZ(0)) + \ln(U(0)) + 1 \le 0.$$

This is reflected in our assumption (6.3).

As a conclusion of this analysis, let us perform the estimate of the behavior of W and U in terms of t when $t \to T$ for the orbits in $\mathbf{F}$ for which $Z \to 0$. Since $f(0) = 0$, the second differential equation in system (6.11) becomes

$$U' = \ln U + o(1) \equiv \ln U \left[1 + O(|\ln U|^{-1})\right],$$

with the end condition $U(T) = 0$. Using the asymptotic expansion of the Schlomich function, we have

$$U(t) = (T-t)|\ln(T-t)|[1 + O(|\ln(T-t)|^{-1})].$$

The first differential equation in system (6.11) then becomes

$$W' = \frac{W}{(t-T)\ln(T-t)}[1 + O(|\ln(T-t)|^{-1})].$$

This differential equation can be easily solved to give the following: there exists $C > 0$ such that for all $t \in [T-\delta, T)$, δ small,

$$W(t) \le C\,|\ln(T-t)|. \tag{6.16}$$

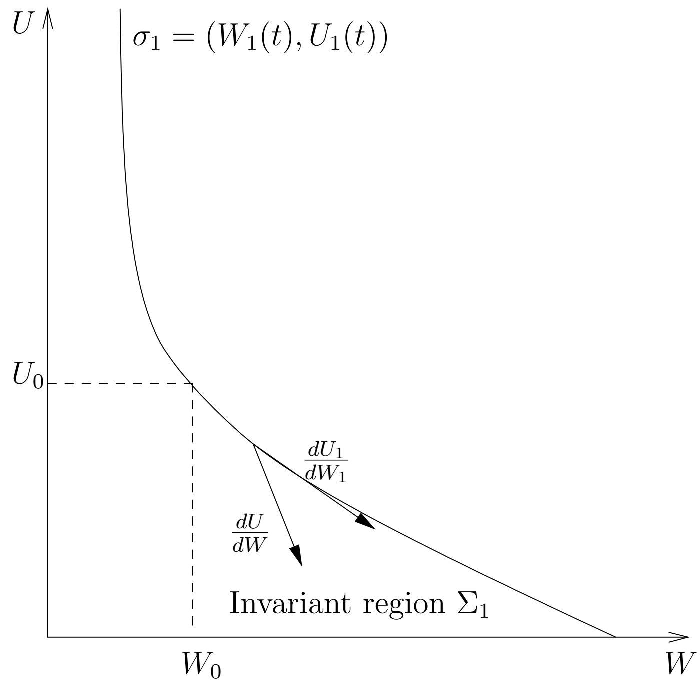

FIG. 2. *Comparison of orbits.*

Step 3. *Comparison and asymptotic estimates.* We proceed now to compare the bounds for the solution u which satisfy the system (6.9), (6.10) with the solutions of the differential system (6.11), whose behavior we have just analyzed. In order to use notation that distinguishes the systems, we use a subscript 1 for the latter. Thus, we still denote by $W(t)$, $U(t)$ the bounds introduced in Step 1, which define a curve σ in the WU-plane. Furthermore, we denote by $\sigma_1 = (W_1(t), U_1(t))$ an orbit solution of (6.11), which quenches in finite time. Let us assume that they have the same initial data $W(0), U(0) > 0$ lying somewhere in the region Γ of the WU-plane. We will show that the region Σ_1 in the WU-plane contained between the orbit σ_1, the W axis, and the vertical line $W = W_0$ is an *invariant region* for the forward evolution of $(W(t), U(t))$ (see Figure 2). Since $U(t)$ is monotone nonincreasing and $W(t)$ is monotone increasing, the only way the orbit (W, U) can escape Σ_1 is through the upper wall given by the orbit of (6.11), which we can write as a decreasing function

$$U_1 = U_1(W_1), \quad W_1 > 0.$$

Now, at any such escape point we have $U = U_1$, $W = W_1$, and besides, $dU_1/dt < 0$, $dW_1/dt > 0$, so that

$$\frac{dU}{dW} < \frac{dU_1}{dW_1},$$

which means that this escape is impossible. We conclude that the curve $U = U(W)$ lies below the orbit $U_1 = U_1(W_1)$; in other words,

$$0 \le U(W) \le U_1(W), \;\; W \ge W(0).$$

Since the functions are monotone, this can also be written $0 \le W(U) \le W_1(U)$, $0 < U < U(0)$. Substituting this estimate from above into the inequality for $U(t)$, we get

$$\frac{dU}{dt} < f(cUW) + \ln(U) \le f(cUW_1(U)) + \ln(U)$$

for all small $U > 0$. Since by the assumption, the curve $\{(U, W_1(U)),\ 0 < U \le U(0)\}$ is contained in the invariant region and hence $Z = UW_1(U) \to 0$ as $U \to 0$, we have that

$$U' < \ln(U) + o(1), \quad U \to 0. \tag{6.17}$$

Integrating this inequality over $(t, T),\ t \approx T$, we obtain

$$U(t) > (T-t)|\ln(T-t)|[1 + O(|\ln(T-t)|^{-1})].$$

Combining this lower bound with that proved in Proposition 6.1, we obtain (6.4). The desired estimate (6.5) for $W(t)$ then follows from (6.9) with the function $U(t)$ already defined by (6.4). In order to end the proof of the theorem, we only have to check that the initial bounds of our solution $(W(0), U(0))$ lie in the region $\mathbf{F}$ where the orbit of system (6.11) satisfies $Z \to 0$. This is the object of condition (6.3). ∎

7. Behavior of the profile near the quenching time. We suppose that the solution of (1.1) quenches in finite time T. We are interested in the behavior of the profile u near the quenching time. For this purpose, we use the results on the stability of the ω-limit sets of perturbed infinite-dimensional dynamical systems [14], [16].

We first rescale (1.1) in order to obtain from the estimates (6.4), (6.5) new time-independent estimates. Then we will show that the rescaled equation is an exponentially small perturbation of a Hamilton–Jacobi equation and we will be ready to use stability theory. In what follows, we suppose that the assumptions on f and u_0 made in Theorem 6.1 are satisfied. As in section 5, we assume that $0 < u(x,t) \le 1/2$ in Q_T.

7.1. Rescaling of (1.1). We first change the unknown u in (1.1) so as to make the estimate (6.5) time-independent. We make the change of the dependent variables

$$u = G(v), \tag{7.1}$$

where the function G is defined implicitly by means of the formula

$$z = \int_{-\ln G(z)}^{\infty} e^{-s}\,\frac{ds}{s}, \quad z > 0.$$

The function G maps $\mathbb{R}^+$ onto $[0,1]$ and it is the solution of the singular ODE

$$G' = -\ln G, \quad z > 0; \qquad G(z) \to 0 \quad \text{as } z \to 0.$$

Notice that

$$G(z) = -z\,\ln(z)\left[1 + O\left(\frac{\ln|\ln z|}{\ln z}\right)\right] \quad \text{for small} \ \ z > 0,$$

and that G is a diffeomorphism from $\mathbb{R}^+$ onto $[0,1)$. We denote its inverse by g. Then

$$g(v) = -\frac{v}{\ln v}(1 + o(1)) \quad \text{as} \ \ v \to 0.$$

In terms of v (1.1) becomes

$$v_t = -\frac{1}{2}\, G'(v)\,(v_x)^2 + \frac{f\left(c\,G'(v)\left(G(v)\,v_{xx} - v_x^2\right)\right)}{G'(v)} \;-1 \quad \text{in} \quad Q_T, \tag{7.2}$$

with $v_x(0,t) = v_x(l,t) = 0,\ \ t \in (0,T);\ v(x,0) = g(u_0(x)) \equiv v_0(x),\ \ x \in (0,l)$. Under the assumptions of Theorem 6.1, the estimates (6.4) and (6.5) become

$$\min_{x\in(0,l)} v(x,t) \equiv v(0,t) = (T-t)\left[1 + O\left(\frac{\ln|\ln(T-t)|}{|\ln(T-t)|}\right)\right], \tag{7.3}$$

$$v_{xx}(x,t) \le C, \quad (x,t) \in [0,l] \times [T-\delta, T). \tag{7.4}$$

Recall that as $u_0'(x) \ge 0$, we have $v_x(x,t) \ge 0$ and moreover the inequality sign is strict in Q_T. At this stage, the second manipulation is to do a rescaling of the two variables x,t in order to obtain new estimates independent of time. Thus we will use the change of variables which has been used in the study of a perturbed Hamilton–Jacobi equation in [16]. For $x \in [0,l]$, $t \in [0,T)$, let us denote

$$\xi = \frac{x}{\sqrt{T-t}}\,,\ \tau = -\ln(T-t)\,,\ v(x,t) = e^{-\tau}\theta(\xi,\tau).$$

For convenience, we denote $(\cdot) = e^{-\tau}\theta(\xi,\tau)$. The problem (7.2) becomes

$$\theta_\tau = \mathbf{H}(\theta) + \mathbf{P}(\theta,\tau) \quad \text{in } \tilde{Q}_T = \{\xi \in (0, le^{\tau/2}),\ \ \tau \in (\tau_0,\infty)\},\ \ \tau_0 = -\ln T, \tag{7.5}$$

and $\theta_\xi(0,\tau) = \theta_\xi(le^{\tau/2},\tau) = 0,\quad \tau \in (\tau_0,\infty);\ \ \theta(\xi,\tau_0) = T^{-1}v_0(\xi\sqrt{T}) \equiv \theta_0(\xi),\ \ \xi \in (0, l/\sqrt{T})$, where the operators $\mathbf{H}$ and $\mathbf{P}$ are defined by

$$\mathbf{H}(\theta) = -\frac{1}{2}\,\xi\,\theta_\xi + \theta - 1,$$

$$\mathbf{P}(\theta,\tau) = -\frac{1}{2} G'(\cdot)\, e^{-\tau}\,\theta_\xi^2 + \frac{1}{G'(\cdot)}[f(cG'(\cdot)(G(\cdot)\theta_{\xi\xi} - e^{-\tau}\theta_\xi^2))].$$

From the definitions of the rescaled variables ξ, τ, and θ, the problem of the asymptotic behavior of $v(x,t)$ near the quenching time T and the quenching point $x_q = 0$ reduces to the study of the rescaled solution $\theta(\xi,\tau)$ in a neighborhood of $\xi = 0$ as $\tau \to \infty$.

The dynamical system (7.5) is a perturbed first-order Hamilton–Jacobi equation

$$h_\tau = \mathbf{H}(h), \quad \tau > 0.$$

The passage to the limit $\tau \to \infty$ in (7.5) is nontrivial since the perturbation term $\mathbf{P}(\theta,\tau)$, to be proved asymptotically small for $\tau \gg 1$, is a *second-order* perturbation, so that this asymptotic problem falls in with the scope of the so-called singularly perturbed infinite-dimensional dynamical systems. The standard methods of the known asymptotic theory do not apply to such a class of problems. Our analysis is based on the stability theorem from [14] specially designed to cover such singularly perturbed systems.

Estimates (7.3) and (7.4) now take the form

$$\min_{\xi\in[0,l\exp(\frac{\tau}{2})]} \theta(\xi,\tau) \equiv \theta(0,\tau) = 1 + O(\ln\tau/\tau), \tag{7.6}$$

$$\theta_{\xi\xi}(\xi,\tau) \le C, \quad \tau \gg 1. \tag{7.7}$$

By the strong maximum principle $\theta_\xi(\xi,\tau) > 0$ for $\xi \in (0, le^{\tau/2})$, $\tau > \tau_0$. It then follows from (7.6) and (7.7) that uniformly on any compact subset $\xi \in [0, B]$ for $\tau \gg 1$

$$0 \le \theta_\xi(\xi,\tau) \le C_1 = C_1(B). \tag{7.8}$$

7.2. Interior regularity for (7.5): Bernstein estimates. Using the above estimates, we have that uniformly on every compact subset in ξ and $\tau \gg 1$ there holds

$$G(e^{-\tau}\theta) = \tau\, e^{-\tau}\,\theta\,(1 + O(\ln\tau/\tau)), \quad G'(e^{-\tau}\theta) = \tau(1 + O(\ln\tau/\tau)).$$

Hence, uniformly on compact subsets, (7.5) takes the form

$$\theta_\tau = \mathbf{H}(\theta) + \frac{1}{\tau}(1 + o(1))\, f(\eta) - \frac{1}{2}\,\tau e^{-\tau}(\theta_\xi^2 + o(1)), \tag{7.9}$$

where

$$\eta = c\tau^2 e^{-\tau}(\theta\theta_{\xi\xi} + o(1)).$$

By $o(1)$ we now also denote the nonlinear operators which on sufficiently smooth functions are of order $o(1)$ as $\tau \to \infty$. None of them affects the asymptotic analysis and they do not enter the estimates.

The first-order operator $\mathbf{H}$ in (7.5) is shown to preserve the higher-order regularity. Indeed, the higher-order derivatives $w = D_\xi^k\theta$ for $k \ge 2$ solve the equation

$$w_\tau = -\frac{1}{2}\xi w_\xi + b_k w,$$

with the coefficient $b_k = (2-k)/2 \le 0$, so that by the maximum principle the equations preserve the higher-order inner regularity. We will use this property below. Let us show that the classical Bernstein method applies to the perturbed fully nonlinear equation (7.9). It follows from the structure of the right-hand side that we have to control two different terms, the fully nonlinear one and the first-order quadratic Hamilton–Jacobi term; both are exponentially small as $\tau \to \infty$ on regular orbits. We prove that similar to the case for a quasi-linear parabolic equation considered in section 5 in [16], the Bernstein technique gives uniform interior bounds for higher-order derivatives. We first differentiate (7.9). The first derivative $z = \theta_\xi$, which has been proved to be uniformly bounded on compact subsets, $0 \le z \le C_1$, solves the following linear parabolic equation:

$$z_\tau = \mathbf{H}_1 z + \tau e^{-\tau}[cf'(\eta)(\theta z_{\xi\xi} + zz_\xi + \cdots) - zz_\xi + \cdots], \tag{7.10}$$

where we omit the higher-order operators, which, according to the agreement above, are of the order $o(1)$ as $\tau \to \infty$ on smooth functions, and keep the main two terms only. Here

$$\mathbf{H}_1 z = -\frac{1}{2}\,\xi z_\xi + \frac{1}{2}z.$$

We thus observe that both main exponentially small terms of this equation are balanced in the sense that they have a common time-dependent multiplier $\tau e^{-\tau}$. This makes it possible to apply the Bernstein technique as on page 1125 of [16]. Setting $z = \phi(h)$ with a smooth strictly increasing function $\phi : [0, 1] \to [0, C_1]$ to be determined later and differentiating the resulting equation for h, we obtain for $w = h_\xi$ a quasi-linear parabolic equation of the form

$$w_\tau = \mathbf{H}_2 w + \mathbf{P}_2(w, \tau), \tag{7.11}$$

where

$$\mathbf{H}_2 w = -\frac{1}{2}\xi w_\xi + \frac{1}{2}\left(\left(\frac{\phi}{\phi'}\right)' - 1\right) w, \tag{7.12}$$

and $\mathbf{P}_2$ contains the main nonlinear terms:

$$\mathbf{P}_2(w, \tau) = \tau e^{-\tau}\left[c f'(\eta)\theta \left(\frac{\phi''}{\phi'}\right)' w^3 + \cdots - \phi' w^2 + \cdots \right], \tag{7.13}$$

with

$$\eta = c\tau^2 e^{-\tau}(\theta\phi' w + \cdots).$$

We now prove the interior regularity driven by an exponentially small parabolic perturbation of the first-order equation of the Hamilton–Jacobi type. First of all, from (7.12) due to the maximum principle, one can see that the operator $\mathbf{H}_2$ preserves the interior regularity on smooth solutions provided that the coefficient of the lower-order term w is nonpositive:

$$\left(\frac{\phi}{\phi'}\right)' \le 1 \quad \text{on} \quad [0, 1]. \tag{7.14}$$

Let us show that the function ϕ can be chosen so that the operator (7.13) obeys the maximum principle for large $|w|$, i.e., $\mathbf{P}_2(w, \tau) < 0$ for $|w| \gg 1$ uniformly in $\tau \gg 1$. We impose the standard condition on ϕ

$$\left(\frac{\phi''}{\phi'}\right)' \le -\alpha_1 < 0 \quad \text{on} \quad [0, 1], \tag{7.15}$$

so that the first term in (7.13) has the right monotonicity for large $|w|$. Both conditions, (7.14) and (7.15), are satisfied by the standard parabolic function

$$\phi(h) = C_1 h(h + 1)/2 \quad \text{on} \quad [0, 1]. \tag{7.16}$$

Consider now the multiplier $f'(\eta)$ in the first term in the case of function (1.2). We can set $f'(\eta) \sim 1$ if $\eta \gg 1$ and we get no novelties in the analysis. If $\eta \ll -1$ (i.e., $w \ll -1$), then $f'(\eta) = -(1 + o(1))/\eta$. In this delicate case the first term in (7.13) can be estimated for $w \ll -1$ as follows,

$$c f'(\eta)\theta \left(\frac{\phi''}{\phi'}\right)' w^3 \ge \alpha_2 \tau^{-2} e^{\tau} w^2 (1 + o(1)), \quad \alpha_2 > 0,$$

so that it is a quadratic term w^2 with the coefficient $\sim e^\tau$, i.e., much larger than that in the second term. Since f' is increasing, this makes it possible to prove desired monotonicity of the operator uniformly in $\tau \gg 1$.

Finally, introducing the function $Z = \chi^2(\xi)w^2 \geq 0$, where χ is a standard nonnegative cut-off function, as in [16, sect. 5], we conclude that, as in the Bernstein method for linear parabolic equations, due to the suitable signs of the coefficients, the parabolic differential inequality for Z does not admit large growing solutions. This gives us an interior estimate of the second derivative which is uniform in $\tau \gg 1$. The proof of similar bounds on higher-order derivatives is simpler and is performed as in [16].

Below we summarize the above results.

LEMMA 7.1. *Under the above hypotheses, the rescaled solution of* (7.5) *with function* (1.2) *satisfies*

$$|D_\xi^k \theta| \leq C_k, \quad k = 2, 3, 4, \dots, \tag{7.17}$$

for $\tau \gg 1$ on any compact subset in ξ.

From the above estimate of the second derivative, we have that uniformly on compact subsets the operator $\mathbf{P}$ takes the following linearized quasi-linear form:

$$\mathbf{P}(\theta, \tau) = -\frac{1}{2}\tau e^{-\tau}(1 + o(1))\theta_\xi^2 + \frac{c}{2}\,\tau e^{-\tau}\left(\theta\theta_{\xi\xi} - \frac{1}{\tau}\,\theta_\xi^2 + o(e^{-\tau})\right).$$

Thus, it follows from the estimates (7.6)–(7.8) that (7.5) is an exponentially small perturbation of the first-order Hamilton–Jacobi equation

$$\theta_\tau = \mathbf{H}(\theta) \equiv -\frac{1}{2}\,\xi\,\theta_\xi + \theta - 1. \tag{7.18}$$

We then can apply the stability theory [14], [16] on comparison of the ω-limit sets of two "close" dynamical systems. We recall therefore the following principal result.

THEOREM 7.1. *Consider two dynamical systems:*

$$u_t = A(u), \quad t > 0, \tag{7.19}$$

$$u_t = B(t, u), \quad t > 0. \tag{7.20}$$

Consider a class $\mathbf{S}$ of solutions $u \in \mathbf{C}([0, \infty), X)$ of the dynamical system (7.20) *defined for $t > 0$ whose values are in a complete metric space X with the distance function d. Suppose the following:*

(H1) *The orbits of the dynamical system* (7.19) $\{u(t)\}_{t>0}$ *are relatively compact in X.*

(H2) *The operator B is a small perturbation of the operator A in the following sense: for a given solution $u \in \mathbf{S}$ of* (7.20)*, if for a sequence $\{t_j\} \to \infty$ there holds $u(t + t_j) \to v(t)$ in $L^\infty_{loc}([0, \infty); X)$, then v is a solution of* (7.19).

(H3) *The ω-limit set of* (7.19) *in X defined by $\Omega = \{f \in X \mid \exists\, u \in \mathbf{C}([0, \infty); X)$, a solution of* (7.19)*, such that $\exists\ \{t_j\} \to \infty$ and $u(t_j) \to f\}$ is nonempty, compact, and uniformly stable in the Lyapunov sense: given an $\epsilon > 0\ \exists\ \delta > 0$ such that for any solution $u \in \mathbf{C}([0, \infty); X)$ of* (7.19) *with $d(u(0), \Omega) \leq \delta \Rightarrow d(u(t), \Omega) \leq \epsilon$ for $t > 0$.*

Then the ω-limit set of the dynamical system (7.20) *in the class $\mathbf{S}$ is contained in Ω, and the solutions of* (7.20) *approach Ω uniformly as $t \to \infty$.*

Let us apply here this theorem to (7.5) and (7.18). We have to find a suitable class of solutions and a suitable metric space. In fact, we will see that once the assumption (H3) of the above theorem is valid, the other assumptions will be automatically valid. Thus we first study (7.18).

7.3. Uniform stability for (7.18). In order to prove that the assumption (H3) is verified for (7.18), we will follow the paper [16], in which the authors studied the extinction phenomenon for a quasi-linear heat equation with a strong absorption by a dynamical systems approach. Let us denote $X_\rho = \{i \in \mathbf{C}([0,\infty)),\ i(0) = 1\,,\ \rho\,(i-1) \in L^\infty([0,\infty))\}$, where we defined the weight function by $\rho(\xi) = \xi^{-2}$. We thus define the distance $d_\rho(i_1, i_2) = \sup_{\xi>0} \rho(\xi)|i_1(\xi) - i_2(\xi)|$ for $i_1\,,\ i_2 \in X_\rho$. The set X_ρ endowed with the distance d_ρ is a complete metric space. Let $\mathbf{C}_\rho$ be the set of function $i \in \mathbf{C}([0,\infty))$ differentiable at the $\xi = 0$ whose seminorm $|i|_\rho = \sup_{\xi>0}\rho(\xi)|i(\xi) - i(0) - i'(0)\xi|$ is finite. The space $\mathbf{C}_\rho$ is a Banach space endowed with the norm defined by $\|i\|_\rho = |i(0)| + |i'(0)| + |i|_\rho$. X_ρ is a closed subset of $\mathbf{C}_\rho$.

Let us now study the Ω-limit set of the Hamilton–Jacobi equation

$$h_\tau = -\frac{1}{2}\,\xi\, h_\xi + h - 1, \quad \tau > 0; \quad h(\xi, 0) = i(\xi). \tag{7.21}$$

If θ is a solution of (7.5), from the two estimates (7.6), (7.7) and $\theta_\xi(0,\tau) = 0$, the function $i \in \mathbf{C}^2(\mathbb{R})$ verifies

$$i(0) = 1\,,\ i'(0) = 0\,,\ i'(\xi) \geq 0\,,\ i''(\xi) \leq C \quad \text{for} \quad \xi > 0 \quad \text{with } C > 0. \tag{7.22}$$

Thus $i \in X_\rho$ and we define the class $\mathbf{S}$ as the functions which satisfy the properties (7.22). Let us note that the class $\mathbf{S}$ is a closed bounded subset of X_ρ. Moreover from the regularity result (7.8), i' is bounded on every compact in ξ of $[0,\infty)$. Thus by the Ascoli theorem, the class $\mathbf{S}$ is compact in X_ρ. Let us show that the ω-limit set of (7.21) is a closed subset of the class $\mathbf{S}$. Let us perform the change of variables $x = \xi e^{-\tau/2}$ and the change of unknown $v = -\log(1-h)$. Then we obtain the problem

$$v_\tau = -1, \quad \tau > 0; \quad v(x, 0) = -\log(1 - i(x)).$$

Thus $v(x,\tau) = -\log(1-i(x)) - \tau$. Going back to the variable (ξ,τ) and the unknown h, we deduce that

$$h(\xi,\tau) = 1 - e^\tau + e^\tau\, i(\xi e^{-\tau/2}), \tag{7.23}$$

and we obtain the following result.

LEMMA 7.2. *If the function $i \in \mathbf{S}$, then $h(\cdot,\tau) \in \mathbf{S}$ for $\tau > 0$.*

Proof. Let $i \in \mathbf{S}$. From the explicit form of h given by (7.23) we have (i) $h(0,\tau) \equiv 1$; (ii) $h_\xi(\xi,\tau) = e^{\tau/2}\, i'(\xi e^{-\tau/2})$. Thus $h_\xi(\xi,\tau) \geq 0\,, h_\xi(0,\tau) = 0$; (iii) $h_{\xi\xi}(\xi,\tau) = i''(\xi e^{-\tau/2})$. Then $h_{\xi\xi}(\xi,\tau) \leq C$; (iv) $\xi^{-2}(1 - h(\xi,\tau)) = x^{-2}(1 - i(x))$, if we denote $x = \xi e^{-\tau/2}$. Thus $\rho\,(1-h) \in L^\infty([0,\infty))$. These complete the proof. ∎

We can now prove that assumption (H3) of Theorem 7.1 is verified.

PROPOSITION 7.1. *The ω-limit set Ω of* (7.18) *is nonempty, compact, and uniformly stable in the Lyapunov sense.*

Proof. Ω is the set of stationary solutions of (7.18). Thus $i \in \Omega \iff \exists\, a \in \mathbb{R}\,,\ i(\xi) = 1 + a\,\xi^2$. From the definition of the weight ρ, Ω is a nonempty closed subset of X_ρ. From Lemma 7.2 we deduce that $h(\cdot,\tau) \in \mathbf{S}$ for $\tau > 0$. We then have that $\Omega \subset \mathbf{S}$ and therefore from the definition of the class $\mathbf{S}$, $i \in \Omega$ implies $i(\xi) = 1 + a\,\xi^2$. But as $\mathbf{S}$ is compact and Ω is a closed subset of $\mathbf{S}$, Ω is a nonempty compact subset of X_ρ.

Let us prove that Ω is uniformly stable in the Lyapunov sense. Let $i \in \mathbf{S}$ be such that $d_\rho(i,\Omega) \leq \epsilon$. As Ω is closed, there exists $a \geq 0$ and $i_a(\xi) = 1 + a\,\xi^2 \in$

Ω, $d_\rho(i,i_a) = d_\rho(i,\Omega)$. From the explicit formula of h, (7.23), we have for $\xi > 0$, $\tau > 0$

$$\frac{1}{\xi^2}[h(\xi,\tau) - i_a(\xi)] = \frac{e^\tau}{\xi^2}[-1 - a\xi^2 e^\tau + i(\xi e^{-\tau/2})] \equiv \frac{1}{x^2}[i(x) - i_a(x)],$$

where $x = \xi\, e^{-\tau/2}$. Therefore, $d_\rho(h(\cdot,\tau), i_a) = d_\rho(i,i_a) = d_\rho(i,\Omega) \le \epsilon$ for all $\tau > 0$. Thus Ω is uniformly stable in the Lyapunov sense. ∎

The assumption (H3) of Theorem 7.1 is verified. Finally, we note that assumption (H1) of Theorem 7.1 is verified and the orbits of the dynamical system (7.5) are relatively compact in X_ρ due to the Bernstein estimates; see Lemma 7.1.

7.4. Asymptotic profile near the quenching time. We now prove that the assumption (H2) of Theorem 7.1 is verified. We show that (7.5) is a small perturbation of (7.18). Let us suppose that u_0 and f verify the assumptions of Theorem 6.1. In this case after the change of unknown and the rescaling, we have that the function θ_0 given in (7.5) is in $\mathbf{C}_\rho$. Let us consider the ω-limit set of the orbit $\theta(\cdot,\tau)$ satisfying the perturbed Hamilton–Jacobi equation (7.5):

$$\omega(\theta_0) = \{i \in \mathbf{C}_\rho \,|\, \exists\ \{\tau_j\} \to \infty \ \text{ such that } \ \theta(\tau_j) \to i \ \text{ in } \ L^\infty_{loc}([0,\infty); X_\rho)\}\,.$$

Given a sequence $\{\tau_j\} \to \infty$, from estimates (7.6)–(7.8), and the Bernstein estimates, by passing to the limit in (7.5), we conclude that $\theta(\cdot, \tau_j{+}\tau) \to h(\cdot,\tau)$ in $L^\infty_{loc}([0,\infty); X_\rho)$, where the function $h(\cdot,\tau)$ is a solution of (7.18) with initial data $i \in \omega(\theta_0)$. Thus assumption (H2) of the Theorem 7.1 is valid. We can now apply Theorem 7.1.

THEOREM 7.2. *Let f and U verify the assumptions of Theorem* 6.1*. Then there exists a unique finite $a \ge 0$ such that $\omega(\theta_0) = i_a$, i.e., $\theta(\xi,\tau) \to i_a(\xi) = 1 + a\,\xi^2$ as $\tau \to \infty$ uniformly on compact subsets.*

Proof. Theorem 7.1 says that $\omega(\theta_0) \subseteq \Omega = \{i_a, a \ge 0\}$, where Ω is the omega-limit set of (7.18). The finiteness of possible a's follows from the uniform upper estimate $\theta_{\xi\xi} \le C$ whence $a \le C/2$. The uniqueness of the profile $i_a \in \omega(\theta_0)$ follows from the ODE for the second derivative at the origin $\theta_{\xi\xi}(0,\tau)$ and a uniform in $\tau \gg 1$ estimate of the fourth derivative at $\xi = 0$; see a similar proof of Proposition 5.6 in [16]. ∎

From the definitions of ξ, τ, θ, and v, the function $v(x,\tau)$ tends to the function $v_{lim}(x,\tau) = T - t + a\,x^2$. This observation makes it possible to derive the *final-time profile* $u(x,T^-)$. By performing the change of unknown defined by G, the solution $u(x,t)$ of (1.1) is expected to tend as $t \to T^-$ to the function

$$u_{lim}(x,T) = G(v_{lim}(x,T))(1+o(1)) \equiv 2a\,x^2|\ln x|(1+o(1)) \quad \text{as} \ \ x \to 0. \tag{7.24}$$

THEOREM 7.3. *Under the assumptions of Theorem* 7.2 *and the convexity assumption* (5.3)*, there holds the parameter $a = a(u_0) > 0$ and then the final-time profile is given by* (7.24).

Proof. The positivity of a follows from Proposition 5.3. The proof of (7.24) is based on a compactness argument in the extension of the asymptotic behavior on compact subsets in the self-similar variable ξ to the behavior for small $x > 0$. See section 8 in [19]. ∎

8. Numerical verification. After performing an analytic study of the asymptotic profile, we now verify the shape of the asymptotic profiles by solving (1.1) numerically. We thus consider the detonation problem in ducts (1.4). The function $f(s) = \ln[(e^s - 1)/s]$ is known to satisfy the assumptions of Theorem 6.1 provided

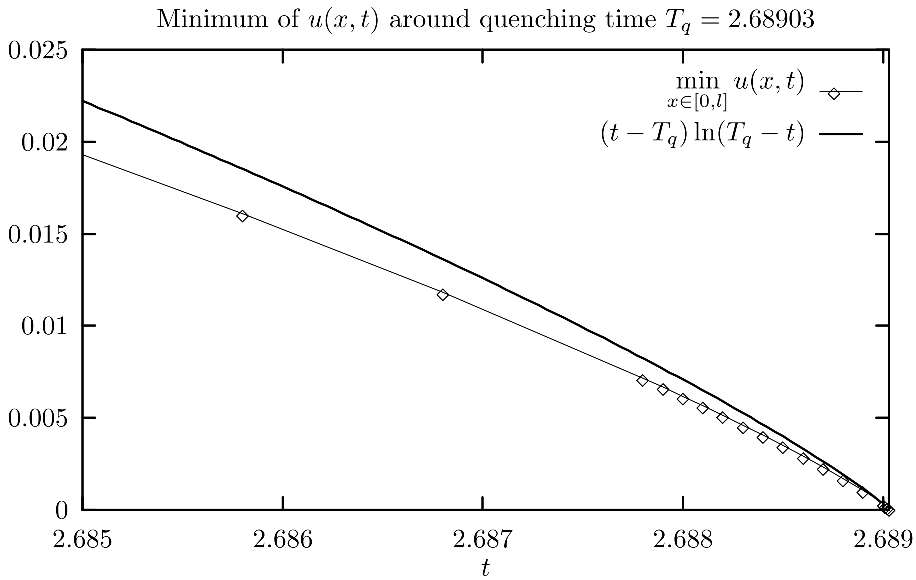

FIG. 3. *Minimum of u near the quenching time $T = 2.68903$.*

$c \in (0,1)$. Therefore, if we choose initial data u_0 satisfying the assumption of Theorem 6.1, we obtain that estimates (6.4), (6.5) hold and that the asymptotic profiles of u near the quenching time in both variables ξ and x are as given by Theorem 7.2 and Theorem 7.3.

We solve numerically the problem by the method of lines; see [32]. This method is an Euler method for the time derivative and a finite-difference method for the space derivatives. In our case, we have to use the Newton method to solve the discrete equation, since (1.4) is nonlinear. We have used the Fortran program MOLCH of the scientific library IMSL [30], with a relative precision of 10^{-6}.

The parameters are $c = 0.268$ and $l = 1.1\,\pi\,\sqrt{c/2}$, since we know that $l_0 = \pi\,\sqrt{c/2}$ is a bifurcation parameter for the stationary solution [4].

The initial conditions are close to the stationary state $u_0 = 1$: we have chosen $U(x) = 1 - 0.1\cos(\pi\, x)$. In this case the solution u quenches "numerically" at the time $T = 2.68903$ at the single point $x = 0$.

In Figure 3, we present the graph of the minimum in x, $u(0,t)$, of the solution $u(x,t)$ in the neighborhood of the quenching time T and the graph $(t-T)\,\ln(T-t)$ versus t. These two curves are in good agreement near the quenching time $T = 2.68903$. We are not able to verify the estimate (6.5) since the approximation of the second derivative by the finite difference method is not accurate enough. But from estimates (6.4), (6.5), since $f(0) = 0$, $f'(0) > 0$, we obtain the following estimate:

$$u_t(0,t) = \ln(T-t)(1+o(1)). \tag{8.1}$$

In Figure 4, we present $u_t(0,t)$ versus $\ln(T-t)$, and we obtain a straight line whose slope is close to 1. Finally, in Figure 5 we present the profile $\theta(\xi,\tau) - 1$ versus ξ^2 in a neighborhood of the quenching time T. We see near the point $\xi = 0$ a straight line whose slope is strictly positive.

9. No continuation beyond singularity exists. The construction of the unique proper solution beyond the quenching singularity, for $t > T$, is performed

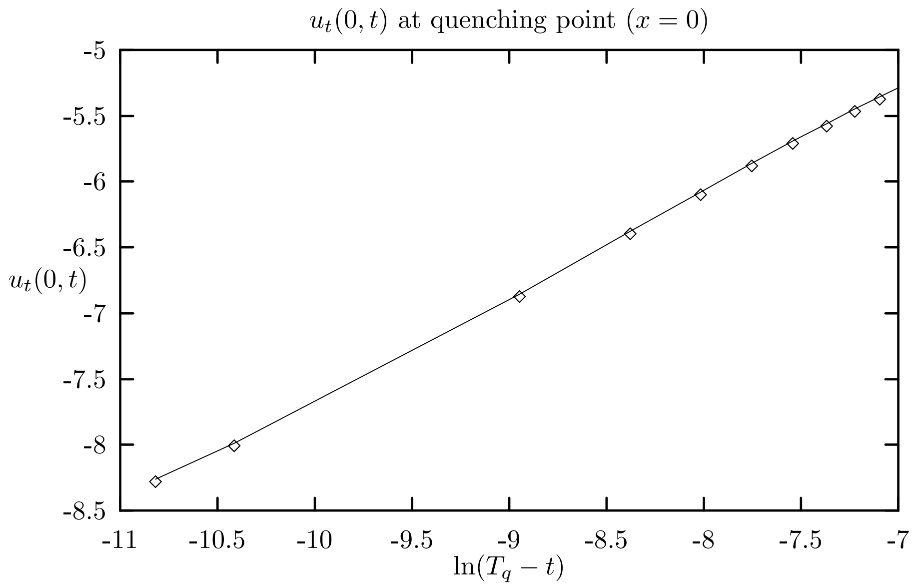

FIG. 4. $u_t(0,t)$ *near the quenching time* $T = 2.68903$ *versus* $\ln(T - t)$.

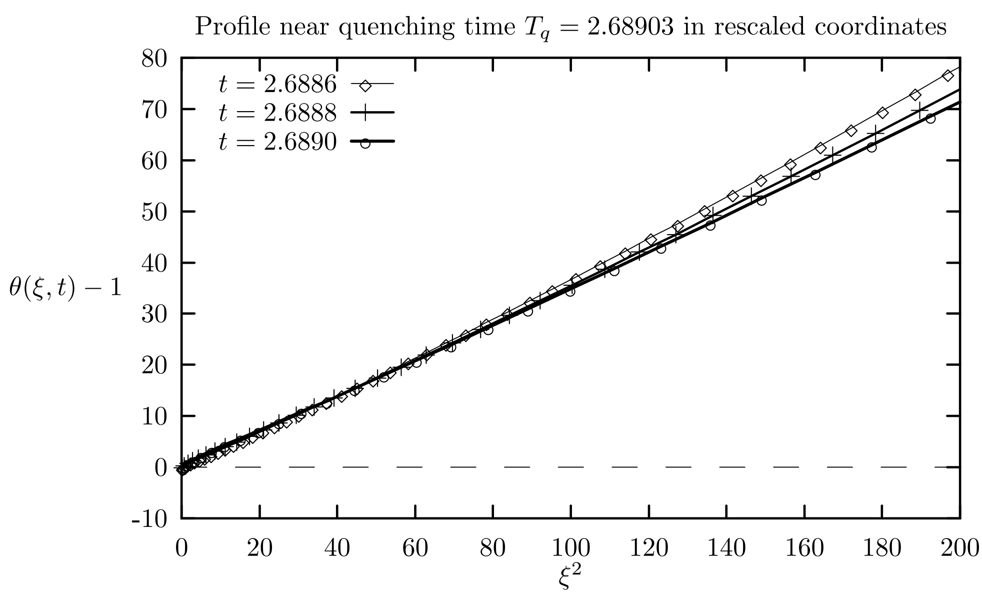

FIG. 5. $\theta(\xi,\tau) - 1$ *near the quenching time* $T = 2.68903$ *versus* ξ^2.

by the extended semigroup theory; see section 2 in [18]. We briefly explain such a construction. We assume, as usual, that $0 < u_0 \le 1/2$, hence $0 < u \le 1/2$ in Q_T. Consider the truncated (nonsingular) equations

$$u_t + \frac{1}{2}(u_x)^2 = f(cuu_{xx}) + \ln u \, \frac{nu^4}{1 + nu^4}, \quad n = 2, 3, \dots, \tag{9.1}$$

with the same initial and boundary conditions. The truncated absorption term $g_n(u) \in C^3([0,1/2])$ is Lipschitz continuous on $[0,1/2]$ and $g_n(0) = 0$. Therefore, the solutions do not reach the singular level $\{u = 0\}$ and for all $n \geq 1$ there exists a unique classical global solution $0 < u_n \leq 1/2$ in Q; see section 2 in [18]. Since the sequence $\{g_n(u)\} \to \ln u$ as $n \to \infty$ uniformly on any interval $[\delta, 1/2]$ with $\delta > 0$ and it is monotone decreasing on $[0,1/2]$, the same is true for the solution sequence $\{u_n\}$ in Q. This follows from the maximum principle. The global *proper* solution is then defined as

$$u = \lim_{n\to\infty} u_n = T_t u_0,$$

which is well defined everywhere in Q. T_t is called the *limit semigroup*. It follows from the construction that $u(x,t)$ is the *maximal* solution which does not depend on the type of the nonincreasing truncation. See [18] for further details.

We now prove the main result.

THEOREM 9.1. *The quenching singularity in the problem* (1.1), (1.2) *is complete in the following sense. Let $t = T < \infty$ be the quenching time of a solution in Q_T. Then its proper extension for $t > T$ is entirely singular:*

$$u \equiv 0 \quad \textit{for} \quad t > T. \tag{9.2}$$

This means that any monotone decreasing approximating sequence $\{u_n\}$ of classical nonsingular solutions satisfies

$$u_n(x,t) \to 0 \quad \text{as } n \to \infty \text{ for any } t > T,\ x \in (-l,l).$$

In other words, since the maximal solution is identically zero beyond quenching, then no other nontrivial solution understood in any weak or mild sense exists in this problem. Note that the theorem states that the limit semigroup T_t is essentially discontinuous at $t = T$.

Proof. As in [17], we apply the technique of intersection comparison with the family $B = \{\theta(x - \lambda t + a),\ \lambda, a \in \mathbb{R}\}$ of traveling wave (TW) solutions. Substituting

$$V(x,t) = \theta(\xi), \quad \xi = x - \lambda t + a,$$

into the equation, we obtain the ODE

$$f(c\theta\theta'') = \frac{1}{2}(\theta')^2 - \lambda\theta - \ln\theta. \tag{9.3}$$

The main result of [17] stated for a general quasi-linear reaction-diffusion equation

$$u_t = (\phi(u))_{xx} \pm f(u)$$

says that the necessary and sufficient condition of *incomplete* singularity (i.e., the existence of a nontrivial continuation for $t > T$) is the existence of a singular TW-profile θ which starts from the singular level. Since the Sturmian argument of zero set analysis applies to classical solutions of fully nonlinear parabolic equations of the type (1.1) via a standard linearization procedure, we now use this technique in the detonation problem (1.1), (1.2).

We first prove nonexistence of a singular TW.

LEMMA 9.1. *For any fixed* $\lambda \in \mathbb{R}$, (9.3) *admits no singular continuous solution* $\theta \not\equiv 0$ *starting from the singular level:*

$$\theta(0) = 0. \tag{9.4}$$

Proof. Setting $\theta' = Y$, we reduce (9.3) to the first-order equation

$$c\theta Y \frac{dY}{d\theta} = f^{-1}\left(\frac{1}{2}Y^2 - \lambda Y - \ln\theta\right), \quad \theta > 0. \tag{9.5}$$

We are looking for an integral curve $\gamma = (\theta, Y(\theta))$ defined on an arbitrarily small interval $\theta \in (0, \delta]$. It follows from the phase portrait of (9.5) that $Y dY/d\theta > 0$ for $\theta > 0$ small. Hence, such a solution $Y(\theta)$ is uniformly bounded: $|Y| \le C_0$ on $(0, \delta]$. Then we get as $\theta \to 0$

$$c\theta Y \frac{dY}{d\theta} = f^{-1}(-\ln\theta + O(1)) = -\ln\theta\,(1 + o(1)).$$

Integrating this equation, we obtain

$$Y^2 = -\frac{1}{c}\ln^2\theta\,(1 + o(1)) \to -\infty \quad \text{as} \quad \theta \to 0.$$

This contradiction proves the result. ∎

Therefore, in order to apply Theorem 4.1 in [17] on complete singularities, we need only to check the "steepness" property of TWs situated near singularity. Namely, we fix $0 < \epsilon \ll 1$ and consider for (9.3) the problem on the orbit starting from the ϵ-neighborhood of the singular level $\{\theta = 0\}$:

$$\theta(0) = \epsilon, \quad \theta'(0) = 0.$$

It reduces to the study of the integral curve of (9.5) starting at the point $(\epsilon, 0)$. Fix $|\lambda| \gg 1$ and $\theta_0 \in (0, 1/2]$. It follows from the general structure of the phase portrait of (9.5) that as $\epsilon \to 0$ there holds $|Y| \to \infty$ uniformly on any level $\{\theta = \theta_* \in [\theta_0, 1/2]\}$. Since $Y = \theta' \equiv V_x$, this means that the TWs satisfy as $\epsilon \to 0$

$$|V_x|_{V=\theta_*} \gg 1 \quad \text{uniformly for} \quad \theta_* \in [\theta_0, 1/2].$$

This is the required condition which means that, given smooth positive initial data $u_0(x) \in C^2([0, l])$, $0 < u_0 \le 1/2$, the number of intersections of the smooth initial functions u_0 and the steep TW-profile $V(x, 0) \equiv \theta(x + a)$ satisfies

$$J(0, V) \le 2 \quad \text{for all} \quad a \in \mathbb{R}.$$

Then using the proof of Theorem 4.1 in [17], by passing to the limit $n \to \infty$, gives that at $t = T^+$ the singular interfaces of the proper (maximal) solution $u(x, t)$ propagate in both directions with a speed larger than $|\lambda|$. Since $|\lambda|$ can be chosen arbitrarily large, this implies the infinite speed of the interface propagation for the proper solution at $t = T^+$, whence the result (9.2). ∎

We now compare the above conclusions with the results on complete/incomplete singularities for quasi-linear parabolic equations. As was proved in section 7, we have that $uu_{xx} \to 0$ as $t \to T^-$ in a small neighborhood of the quenching point. Therefore, linearizing the fully nonlinear term $f(s) = s/2 + O(s^2)$ for $s \approx 0$, one can expect that the quenching singularity might be described by the *linearized quasi-linear* equation

$$u_t = \frac{c}{2}\, u u_{xx} - \frac{1}{2}\,(u_x)^2 + \ln u \quad \text{in} \;\; \{0 < u \ll 1\}.$$

Setting $u = v^{-c}$ (then v blows up when u quenches), we then arrive at the fast diffusion equation with a special reaction term

$$v_t = \alpha (v^m)_{xx} + v^{2-m} \ln v \quad \text{in} \;\; \{v \gg 1\}, \tag{9.6}$$

where $m = 1 - c > 0$ and $\alpha = c/2(1-c)$. This is a logarithmically perturbed (by the factor $\ln v$ in the reaction term) quasi-linear heat equation from combustion theory

$$v_t = (v^m)_{xx} + v^p, \quad m > 0, \;\; p > 1. \tag{9.7}$$

The properties of blow-up solutions to (9.7) are well established; see Chapter 4 in [31] and references in [18], [19]. In particular, the exponent $p_* = 2 - m$ in (9.7) given by (9.6) is *critical* relative the complete/incomplete blow-up. Namely, the blow-up is incomplete for all $p \in (1, 2-m]$ and is complete if $p > 2 - m$ [17]. It turns out that in the incomplete range $p \in (1, 2-m]$ singular solutions which blow-up at a finite $t = T$ admit unique minimal extension $u(x,t) \not\equiv \infty$ for $t > T$, and the problem becomes a free-boundary problem with singular interfaces on which $u = \infty$. These singular interfaces exhibit interesting dynamics and essentially nonanalytic regularity [20]. If $p > p_*$ then the minimal proper extension is entirely singular: $u \equiv \infty$ for $t > T$ and blow-up is always complete.

Thus, the linearized equation (9.6) with the critical exponent $p = 2-m$ and a slow-growth but unbounded as $v \to \infty$ logarithmic perturbation in the reaction term shows that the blow-up interfaces must propagate with infinite speed at $t = T^+$ which means complete singular behavior, as already proved in Theorem 9.1. On the other hand, this implies that there exists a slight modification of the detonation equation which destroys the slow-growth factor in the corresponding linearized version. This might give a singular behavior with a nontrivial proper extension beyond the singularity described by a free-boundary problem.

Acknowledgments. The authors acknowledge the hospitality of the École Normale Supérieure de Lyon where this project was started and are grateful to C. Brauner and C. Schmidt-Lainé for fruitful discussions and informations.

REFERENCES

[1] S. Angenent, *Nonlinear analytic semiflows*, Proc. Roy. Soc. Edinburgh Sect. A, 115 (1990), pp. 91–107.

[2] D.G. Aronson and Ph. Bénilan, *Régularité des solutions de l'équation des milieux poreux dans* $\mathbf{R}^n$, C. R. Acad. Sci. Paris Sér. I Math., 288 (1979), pp. 103–105.

[3] G.I. Barenblatt, *Similarity, Self-Similarity, and Intermediate Asymptotics*, Consultants Bureau, Plenum, New York, London, 1979.

[4] C.M. Brauner, S. Gerbi, and Cl. Schmidt-Lainé, *A stability analysis for a fully nonlinear problem in detonation theory*, Appl. Anal., 56 (1995), pp. 351–380.

[5] J.D. Buckmaster, *A theory for triple point spacing in overdriven detonation waves*, Combust. and Flame, 77 (1989), pp. 219–228.

[6] J.D. Buckmaster, J.W. Dold, and C. Schmidt-Lainé, *Pressure spot formation in unstable detonation waves*, in Numerical Combustion. Proceedings of the Third International Conference on Numerical Combustion, Juan les Pins, Antibes, 1989, A. Dervieux and B. Larrouturou, eds., Lecture Notes in Phys. 351, Springer-Verlag, Berlin, 1989, pp. 428–439.

[7] J.D. Buckmaster and G.S.S. Ludford, *The effect of structure on the stability of detonations* I. *Role of induction zone*, in Proceedings of the 21st Symposium on Combustion, The Combustion Institute, Pittsburgh, PA, 1987, pp. 1669–1676.

[8] G. Dong, *Nonlinear Partial Differential Equations of Second Order*, Transl. Math. Monogr. 95, Amer. Math. Soc., Providence, RI, 1991.

[9] W. Fickett and W.D. Davis, *Detonation*, University of California Press, Berkeley, CA, 1979.

[10] A. Friedman, *Partial Differential Equation of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[11] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J., 34 (1985), pp. 425–447.

[12] V.A. Galaktionov, *Dynamical systems of inequalities and nonlinear parabolic equations*, Comm. Partial Differential Equations, 24 (1999), pp. 2191–2236.

[13] V.A. Galaktionov and S.A. Posashkov, *Estimates of localized unbounded solutions of quasilinear parabolic equations*, Differential Equations, 23 (1987), pp. 750–759.

[14] V.A. Galaktionov and J.L. Vazquez, *Asymptotic behaviour of nonlinear parabolic equations with critical exponents. A dynamical systems approach*, J. Funct. Anal., 100 (1991), pp. 435–462.

[15] V.A. Galaktionov and J.L. Vazquez, *Extinction for a quasilinear heat equation with absorption* I. *Technique of intersection comparison*, Comm. Partial Differential Equations, 19 (1994), pp. 1075–1106.

[16] V.A. Galaktionov and J.L. Vazquez, *Extinction for a quasilinear heat equation with absorption* II. *A dynamical systems approach*, Comm. Partial Differential Equations, 19 (1994), pp. 1107–1137.

[17] V.A. Galaktionov and J.L. Vazquez, *Necessary and sufficient conditions for complete blow-up and extinction for one-dimensional heat equations*, Arch. Rational Mech. Anal., 129 (1995), pp. 225-244.

[18] V.A. Galaktionov and J.L. Vazquez, *Continuation of blowup solutions of nonlinear heat equations in several dimensions*, Comm. Pure Appl. Math., 50 (1997), pp. 1–67.

[19] V.A. Galaktionov and J.L. Vazquez, *Blow-up for quasilinear heat equations described by means of nonlinear Hamilton-Jacobi equations*, J. Differential Equations, 127 (1996), pp. 1–40.

[20] V.A. Galaktionov and J.L. Vazquez, *Incomplete blow-up and singular interfaces for quasilinear heat equations*, Comm. Partial Differential Equations, 22 (1997), pp. 1405–1452.

[21] I.S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York, 1980.

[22] A.S. Kalashnikov, *Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations*, Russian Math. Surveys, 42 (1987), pp. 169–222.

[23] H. Kawarada, *On solution of initial boundary value problem* $u_t = u_{xx} + 1/1 - u$, Publ. Res. Inst. Math. Sci., 10 (1975), pp. 729–736.

[24] N.V. Krylov, *Nonlinear Elliptic and Parabolic Equations of the Second Order*, D. Reidel, Dordrecht, The Netherlands, 1987.

[25] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr. 23, Amer. Math. Soc., Providence, RI, 1968.

[26] H.A. Levine, *Quenching and beyond: A survey of recent results*, in Proceedings of the International Conference on Nonlinear Mathematical Problems in Industry, Iwaki Meisei University, Iwaki, Japan, 1992, H. Kawarada et al., eds., GAKUTO Internat. Ser. Math. Sci. Appl. 2, Gakkōtosho, Tokyo, 1993, pp. 501–512.

[27] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progr. Nonlinear Differential Equations Appl. 16, Birkhäuser Verlag, Basel, 1995.

[28] D. Da Prato and A. Lunardi, *Stability, instability and center manifolds theorems for fully nonlinear autonomous parabolic equations in Banach spaces*, Arch. Rational Mech. Anal., 101 (1988), pp. 115–141.

[29] G. Da Prato and P. Grisvard, *Équation d'évolutions abstraites de type parabolique*, Ann. Mat. Pura Appl. (4), 120 (1979), pp. 329–396.

[30] J.R. Rice, *Numerical Methods, Software, and Analysis: IMSL Reference Edition*, McGraw-Hill, New York, 1983.

[31] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, and A.P. Mikhailov, *Blow-up in Quasilinear Parabolic Equations*, Walter de Gruyter, Berlin, New York, 1995.

[32] W.E. Schiesser, *The Numerical Method of Lines. Integration of Partial Differential Equations*, Academic Press, New York, 1991.

[33] F. Williams, *Combustion Theory*, 2nd ed., Benjamin Cummings, Menlo Park, CA, 1985.