A kinetic formulation for a model coupling free surface and pressurised flows in closed pipes

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Abstract

The aim of this paper is to present a kinetic formulation of a model for the coupling of transient free surface and pressurised flows. Firstly, we revisit the system of Saint-Venant equations for free surface flow: we state some properties of Saint-Venant equations, we propose a kinetic formulation and we verify that this kinetic formulation leads to a *Gibbs equilibrium* that minimises (in some general case) an energy and preserves the still water steady state. Secondly, we propose a model for pressurised flows in a Saint-Venant-like conservative formulation. We then propose a kinetic formulation and we verify that this kinetic formulation leads to a *Gibbs equilibrium* that minimises in any case an energy and preserves the still water steady state. Finally, we propose a dual model that couples these two types of flow.

Key words: water transients in pipes, free-surface flows, pressurised flows, Saint-Venant like equations, kinetic formulation

1 Introduction

In this paper we are interesting in flows occurring in closed pipes. Thus it may happen that some parts of the flow are free-surface (this means that only a part of the section of the pipe is filled) and other parts are pressurised (this means that all the section of the pipe is filled: see the figure 1). The phenomenon of transition from free surface to pressurised flow occurs in many situations as storm sewers, waste or supply pipes in hydroelectric installations. It can be

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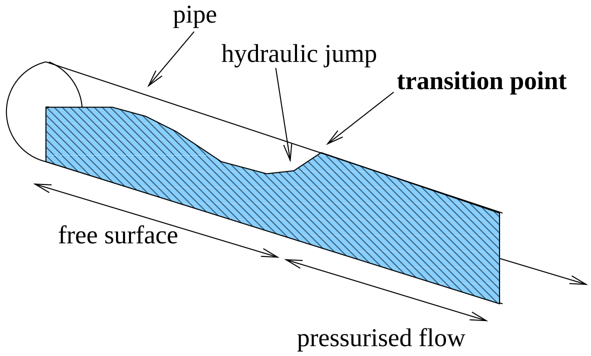

Fig. 1. Mixed flow: free surface and pressurised

induced by sudden changes in the boundary conditions (failure of a pumping station, rapid change of the discharge, blockage of the line etc.). During the transition, the excess pressure rise may damage the pipe and cause related problems as ejection of manhole covers, basement flooding. The simulation of such a phenomenon is thus a major challenge and a great amount of works were devoted to it these last years (see [6],[8],[5] for instance).

The Saint Venant equations, which are written in a conservative form, are usually used to describe free surface flows of water in open channels. As said before, they are also used in the context of mixed flows (i.e. either free surface or pressurised) using the artifice of the Preissman slot [8],[5]. On the other hand, the commonly used model to describe pressurised flows in pipelines is the system of the Allievi equations [8]. This system of 1st order partial differential equations cannot be written under a conservative form since this model is derived by neglecting some acceleration terms. This non conservative formulation is not appropriate for a good approximation of the transition between the two types of flows since we are not able to write conservations of appropriate quantities such as momentum and energy. Then, it appears that a "unified" modelisation with a common set of conservative variables could be of a great interest for the coupling between free-surface and pressurised flows and its numerical simulation could be more effective. In two recent papers [4,2], two of the authors proposed a model for the coupling of free surface and pressurised flows in pipes. They also derived a finite volume scheme to solve numerically this system of partial differential equations with a special treatment of the interface between the two types of flows.

Another approach for the numerical resolution of Saint Venant equations is to use a kinetic formulation. The corresponding scheme appears to have interesting theoretical properties: the scheme preserves the still water steady state and posseses a conservative in-cell entropy inequality. Moreover, this type of numerical approximation leads to an easy implementation.

Recently, Perthame *et al.* [1,7] propose a kinetic scheme for the Saint-Venant equations in rectangular channels with a source term due to the topography.

The aim of this paper is: (i) to propose a kinetic formulation of the Saint-Venant equations for free surface flows in closed pipes, (ii) to construct a kinetic formulation of the model for pressurised flows in closed pipes, (iii) to couple these two formulations to describe mixed flows in closed pipes.

2 Results about Saint-Venant equations in any closed pipes

In this section, we present some properties of the Saint-Venant equations in uniform closed pipes. Then by analogy with Euler equations of compressible gas dynamics, we link the macroscopic Saint-Venant system to a microscopic description of the fluid: it is the kinetic formulation. We state its main properties: the kinetic formulation minimises an energy and preserves the still water steady state.

2.1 Properties of the system of Saint-Venant

The system of Saint-Venant for free surface flows in uniform closed pipes can classically be written as:

$$\partial_t A + \partial_x Q = 0 \tag{1}$$

$$\partial_t Q + \partial_x\left(\frac{Q^2}{A} + gI_1\right) = gA(-\partial_x Z - S_f) \quad . \tag{2}$$

The unknowns are the cross-sectional flow area $A = A(x,t)$, and the discharge $Q = A\,u$ where u is the mean value of the speed over the cross-section in the x-axis direction. The term $g\,I_1$, with $I_1 = \displaystyle\int_0^{h(x,t)} (h-z)\,\sigma(z)\,dz$, arises from the hydrostatic pressure law, where $\sigma(z)$ represents the width of the pipe at the elevation z and $h(x,t)$ is the total water depth (see the figure 2). Let us remark that from the definition of I_1 we have: $I_1(A) = A\,\overline{y}$ and $\dfrac{\partial I_1}{\partial A} = A\dfrac{\partial h}{\partial A}$, where $\overline{y}$ is the distance between the center of mass and the free surface of water (see the figure 2 for the notations). The friction term S_f is assumed to be given by the Manning-Strickler law (see [8]):

$$S_f = K(A)\,u\,\mid u\mid \qquad \text{with} \qquad K(A) = \frac{1}{K_s^2\,R_h(A)^{4/3}} \tag{3}$$

where $K_s > 0$ is the Strickler coefficient, depending on the material, and $R_h(A)$ is the so called hydraulic radius given by $R_h(A) = \dfrac{A}{P_m}$, P_m being the wet perimeter (length of the part of the channel's section in contact with the water).

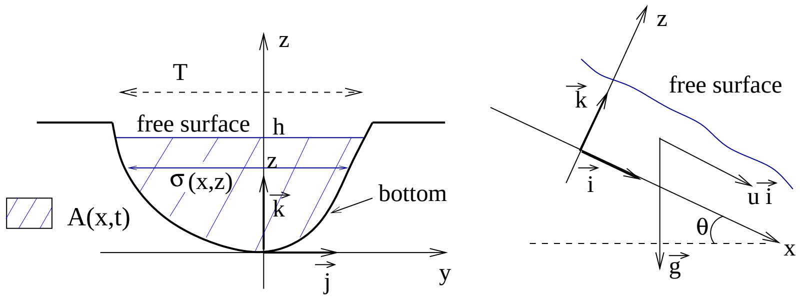

Fig. 2. free surface flow in an open channel

This system can be derived from the incompressible Euler equations by taking mean values in sections orthogonal to the main flow axis. The free surface is advected by the flow and is assumed to be horizontal in the y direction. The distribution of the pressure is supposed to be hydrostatic:

$$P(x, y, z) = P_a + \rho\, g\, (h(x) - z) \tag{4}$$

where P_a is the pressure at the free surface and ρ the density of the water at the pressure P_a (this means that the acceleration of a particle in the plane orthogonal to a streamline is zero). The system (1),(2) writes under the conservative form:

$$\partial_t U + \partial_x F(U) = G(x, U) \tag{5}$$

where the unknown state is $U = (A, Q)^t$.
The flux vector is $F(x, U) = (Q, \dfrac{Q^2}{A} + gI_1)^t$ and the source term writes $G(x, U) = (0, -gA(\partial_x Z + S_f))^t$.
In the sequel, we will suppose, for the sake of simplicity, that the friction term vanishes. This system of partial differential equation is naturally posed for $A(x, t) \geq 0$ and the water wetted area can indeed vanish (flooding zones, dry soils, tidal flat): this fact leads to a theoretical and numerical difficulty since the system loses hyperbolicity when $A(x, t) = 0$. Indeed, we have the following theorem.

Theorem 1 *The system (5) is strictly hyperbolic for $A(x, t) > 0$. It admits a mathematical entropy;*

$$E(A, Q, Z) = \frac{Q^2}{2A} + gAZ + gA(h(A) - \overline{y}) = \frac{Au^2}{2} + gAZ + gAh(A) - gI_1(A) \tag{6}$$

which satisfies the entropy inequality:

$$\partial_t E + \partial_x [u(E + gI_1)] \leq 0 \quad . \tag{7}$$

Proof of theorem 1 Setting $c = \sqrt{g\dfrac{\partial I_1}{\partial A}} = \sqrt{g\,\dfrac{A}{T}}$, the speed of sound, since $DF(U) = \begin{pmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{pmatrix}$ with $u = \dfrac{Q}{A}$ (average velocity along the flow axis), this system is strictly hyperbolic for $A(x,t) > 0$. The eigenvalues are $\lambda = u \pm c$ and the associated right eigenvectors are $r = \begin{pmatrix} 1 \\ u \pm c \end{pmatrix}$. An easy computation leads to the entropy inequality (7). We just recall that for smooth solutions the inequality (7) becomes an equality.

■

Also, the system (5) admits a family of smooth steady states characterized by the relations:

$$Q = Au = C_1 \ , \tag{8}$$

$$\frac{u^2}{2} + g\,h(A) + g\,Z = C_2 \ , \tag{9}$$

where C_1 and C_2 are two arbitrary constants. The quantity $\dfrac{u^2}{2} + g\,h(A) + g\,Z$ is called the total head. Indeed, an easy computation leads to the following partial differential equation for the velocity u:

$$\partial_t u + \partial_x \left(\frac{u^2}{2} + g\,h(A) + g\,Z\right) = 0 \quad . \tag{10}$$

The equations (8), (9) are thus obtained in setting the time derivative to zero in equations (1), (2) and (10).

2.2 The kinetic approach

The work presented in this section is a generalisation to uniform closed pipes of the work of Perthame *et al.* [1,7]. Let us consider a smooth real function χ which has the following properties:

$$\chi(\omega) = \chi(-\omega) \geq 0 \ , \ \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1. \tag{11}$$

We then define the density of particles $\mathcal{M}_{FS}(t,x,\xi)$ (*subscript FS for free surface flow*) by the so-called *Gibbs equilibrium*:

$$\mathcal{M}_{FS}(t,x,\xi) = \mathcal{M}_{FS}(A, \xi - u) = \frac{A}{\sqrt{g\overline{y}}}\chi\left(\frac{\xi - u(t,x)}{\sqrt{g\overline{y}}}\right).$$

These definitions allow to obtain a kinetic representation of the system (5) by the following result.

Theorem 2 *The couple of functions (A,Q) is a strong solution of the system (5) if and only if $\mathcal{M}_{FS}(A,\xi-u)$ satisfies the kinetic equation:*

$$\frac{\partial}{\partial t}\mathcal{M}_{FS} + \xi\cdot\frac{\partial}{\partial x}\mathcal{M}_{FS} - g\frac{\partial}{\partial x}Z\cdot\frac{\partial}{\partial \xi}\mathcal{M}_{FS} = K(t,x,\xi) \tag{12}$$

for some collision term $K(t,x,\xi)$ which satisfies for a.e. (t,x)

$$\int_{\mathbb{R}} K\,d\xi = 0\ ,\ \int_{\mathbb{R}} \xi\,K d\,\xi = 0 \quad . \tag{13}$$

Proof of theorem 2 The proof relies on a very obvious computation. Indeed, the two Eqs. (1),(2) are obtained by taking the moments of the kinetic equation (12) with respect to ξ against 1, ξ and ξ^2 : the righthand side vanishes according to (13) and the lefthand sides coincides exactly thanks to (11). These are consequences of the following relations verified by the microscopic equilibrium:

$$A = \int_{\mathbb{R}} \mathcal{M}_{FS}(\xi)\,d\xi\ , \tag{14}$$

$$Q = \int_{\mathbb{R}} \xi\mathcal{M}_{FS}(\xi)\,d\xi\ , \tag{15}$$

$$g\,I_1(A) + \frac{Q^2}{A} = \int_{\mathbb{R}} \xi^2\mathcal{M}_{FS}(\xi)\,d\xi\ . \tag{16}$$

■

This theorem produces a very useful consequence: the nonlinear Saint-Venant system can be viewed as a simple linear equation on a nonlinear quantity $\mathcal{M}$ for which it is easier to find simple numerical schemes with good theoretical properties: it is this feature which will be exploited to construct a kinetic scheme. For this sake, we characterise the function χ which defines the density of particles $\mathcal{M}(t,x,\xi)$ in its kinetic approach. In particular, we will justify the interpretation of such a density as the microscopic equilibrium of the system: it is the so-called *Gibbs equilibrium*.

Theorem 3 *Let $A(x,t)$ and $Q(x,t)$ be two given functions. Define $k(x,t)$ by $k = g\,h(A) - 2\,g\,\overline{y}(A)$.*

(1) The minimum of the energy:

$$\mathcal{E}(f) = \int_{\mathbb{R}} \left(\frac{\xi^2}{2}f(\xi) + \frac{2}{3}\left(\frac{\pi g\overline{y}}{A}\right)^2 f^3(\xi) + (gZ+k)f(\xi)\right)d\xi \quad ,$$

under the constraints:

$$f \geq 0\,, \int_{\mathbb{R}} f(\xi) d\xi = A\,, \int_{\mathbb{R}} \xi f(\xi) d\xi = Q \quad ,$$

is attained by the function $\mathcal{M}_{FS}(A, \xi - u) = \frac{A}{\sqrt{g\overline{y}}} \chi\left(\frac{\xi - u}{\sqrt{g\overline{y}}}\right)$ *where* χ *is defined by:*

$$\chi(\omega) = \frac{1}{\pi}\sqrt{(1 - \frac{1}{4}\omega^2)_+} \quad . \tag{17}$$

(2) *Moreover, the function* χ *defined by (17) ensures us to have the relation*

$$\mathcal{E}(\mathcal{M}_{FS}) = E(A, Q, Z)$$

if A *and* Q *are solution of Saint-Venant equations (5) and the entropy* E *is defined by (6).*

Proof of theorem 3

(1) Because of the constraints, it is sufficient to minimize the functionnal:

$$\int_{\mathbb{R}} \left(\frac{\xi^2}{2} f(\xi) + \frac{2}{3}\left(\frac{\pi g\overline{y}}{A}\right)^2 f^3(\xi)\right) d\xi.$$

The Euler-Lagrange equation associated to the minisation problem reads:

$$\frac{\xi^2}{2} + 2(\frac{\pi g\overline{y}}{A})^2 f^2 = \lambda + \mu\xi$$

where $\lambda(A, Q)$ and $\mu(A, Q)$ are the Lagrange multipliers. One may easily verify that the function $f = \mathcal{M}_{FS}(A, \xi - u) = \frac{A}{\sqrt{g\overline{y}}}\chi\left(\frac{\xi - u}{\sqrt{g\overline{y}}}\right)$ is a solution of the minimisation problem.

Moreover, as $f \geq 0$, the function $\mathcal{E}(f)$ is strictly convex which ensures as the unicity of the minimum.

(2) Writing $\mathcal{E}(\mathcal{M}_{FS}(A, \xi - u))$ and using the macroscopic representation (14), (15), (16) leads to the above equality. ■

Remark 4 It is the second point of the above theorem that motivates the choice of the quantities $\frac{2}{3}\left(\frac{\pi g\overline{y}}{A}\right)^2$ and k in the formula for the energy $\mathcal{E}$, and the choice of the function χ defined by (17).

We conclude this kinetic formulation of Saint-Venant equations by examining if the above function χ ensures that the Gibbs equilibrium $\mathcal{M}_{FS}$ is solution of the still water steady state, says $u = \frac{Q}{A} = 0$ and $h(A) + Z = \textit{constant}$. The kinetic equation of the still water steady state writes:

$$\xi \cdot \frac{\partial \mathcal{M}_{FS}}{\partial x} - g\frac{\partial Z}{\partial x} \cdot \frac{\partial \mathcal{M}_{FS}}{\partial \xi} = 0 \quad . \tag{18}$$

Remark 5 Setting $u = 0$, $A(x,t) = A(x)$ and $h(A) + Z = constant$ in $\mathcal{M}_{FS}(A, \xi - u))$, and defining the function χ by (17), a tedious computation leads to:

$$\xi \cdot \frac{\partial \mathcal{M}_{FS}}{\partial x} - g\frac{\partial Z}{\partial x} \cdot \frac{\partial \mathcal{M}_{FS}}{\partial \xi} = \frac{1}{8}\frac{\partial A}{\partial x}\frac{3-\omega^2}{\sqrt{(1-\frac{1}{4}\omega^2)_+}}\,\omega(1 - \frac{A}{2T\overline{y}}) \quad .$$

For the case of a rectangular pipe, we have $A = 2T\overline{y}$. Thus defining the function χ by (17) ensures us that the Gibbs equilibrium minimises the energy **and** is solution of the equation of the still water steady state (18) as pointed out by [7]. To obtain these two properties, we have to change the function χ and the definition of the energy $\mathcal{E}$. It is the object of the following result.

Proposition 6 *Let us define:*
$m = \sqrt{\dfrac{2A}{A - \overline{y}T}}$, $J(A) = \int_{-1}^{1}(1 - s^2)^\gamma ds$, $\gamma = \dfrac{3\overline{y}T - A}{2(A - \overline{y}T)}$,
$c(A) = \dfrac{\gamma}{2(1+\gamma)}\dfrac{m^{\frac{2\gamma+1}{\gamma}}J^{\frac{1}{\gamma}}}{A^{\frac{1}{\gamma}}}(g\overline{y})^{1+\frac{1}{2\gamma}}$. *Define the function* χ *by:*

$$\chi(\omega) = \frac{1}{mJ(A)}\left(1 - \frac{\omega^2}{m^2}\right)_+^\gamma$$

and the energy,

$$\mathcal{E}(f) = \int_{\mathbb{R}}\left[\frac{\xi^2}{2}f + c(A)f^{\frac{1}{\gamma}+1} + gf\left(Z + h(A) - \frac{A\overline{y}}{A - \overline{y}T}\right)\right]d\xi \quad ,$$

the Gibbs equilibrium $\mathcal{M}_{FS} = \dfrac{A}{\sqrt{g\overline{y}}}\chi\left(\dfrac{\xi - u}{\sqrt{g\overline{y}}}\right)$ *realises the minimum of the energy* $\mathcal{E}(f)$ *under the same constraints as in Theorem 3, satisfies the relation* $\mathcal{E}(\mathcal{M}_{FS}) = E(A, Q, Z)$ *and is a solution of the still water equation (18).*

Proof of theorem 6 The proof of this result leans on the macroscopic representation (14), (15), (16) and fastidious computations.

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Remark 7 Unfortunately, the energy $\mathcal{E}$ is convex only if $\gamma > 0$ that says $1 < \dfrac{A}{\overline{y}T} < 3$, which is again true for the case of rectangular or trapezoidal pipes or some pipes where T does not tend to 0 as in the circular pipe almost full of water. In the practical computations with the kinetic scheme, the function χ

is chosen in such a way that the integrals are "easy" to compute, e.g. $\chi(\omega) = \dfrac{1}{2\sqrt{3}} \mathbb{1}_{[-\sqrt{3},-\sqrt{3}]}(\omega)$ and very good results are nevertheless observed.

3 Results about pressurised flows in closed pipes

We derived a conservative model for pressurised flows from the 3D system of compressible Euler equations by integration over sections orthogonal to the flow axis. The equation for conservation of mass and the first equation for the conservation of momentum are:

$$\partial_t \rho + \mathrm{div}(\rho \vec{U}) = 0 \tag{19}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \vec{U}) = F_x - \partial_x P \tag{20}$$

with the speed vector $\vec{U} = u\vec{i} + v\vec{j} + w\vec{k} = u\vec{i} + \vec{V}$, where the unit vector $\vec{i}$ is along the main axis, ρ is the density of the water. We use the Boussinesq linearised pressure law (see [8]):

$$P = P_a + \frac{1}{\beta}\left(\frac{\rho}{\rho_0} - 1\right), \tag{21}$$

where ρ_0 is the density at the atmospheric pressure P_a and β the coefficient of compressibility of the water. Exterior strengths $\vec{F}$ are the gravity $\vec{g}$ and the friction $-S_f\,\vec{i}$ with S_f still given by (3). Then equations (19)-(20) become:

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) + \mathrm{div}_{(y,z)}(\rho \vec{V}) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \mathrm{div}_{(y,z)}(\rho u \vec{V}) &= -\rho g(\partial_x Z + S_f) - \frac{\partial_x \rho}{\beta \rho_0} \end{aligned} .$$

Assuming that the pipe is infinitely rigid and has a uniform cross-section A, and taking mean values in sections orthogonal to the main flow axis, we get the following system written in a conservative form for the unknowns $M = \rho A$, $D = \rho Q$:

$$\partial_t(M) + \partial_x(D) = 0 \tag{22}$$

$$\partial_t(M) + \partial_x\left(\frac{D^2}{M} + c^2 \rho M\right) = -\rho g M(\partial_x Z + S_f) \tag{23}$$

where $c = \dfrac{1}{\sqrt{\beta \rho_0}}$ is the speed of sound. A complete derivation of this model, taking into account the deformations of the pipe, and a spatial second order Roe-like finite volume method in a linearly implicit version is presented in [2] (see [3] for the first order implicit scheme). This sytem of partal differential equation is formally close to the Saint-Venant equations (5). To obtain a closer system of partial differential equation, we define an "FS-equivalent" wet area (FS for Free Surface) A_{eq} through the relation:

$$M = \rho\, A_{max} = \rho_0\, A_{eq} \ ,$$

A_{max} being the cross sectional area, and a "FS-equivalent discharge" Q_{eq} by:

$$D = \rho\, Q = \rho_0\, Q_{eq} \quad .$$

Dividing (22)-(23) by ρ_0 we get:

$$\partial_t A_{eq} + \partial_x Q_{eq} = 0 \tag{24}$$

$$\partial_t Q_{eq} + \partial_x \left(\frac{Q_{eq}^2}{A_{eq}} + c^2\, A_{eq} \right) = -g\, A_{eq}\, (\partial_x Z + S_f) \quad . \tag{25}$$

Let us hereby omit the subscript *eq*. The system (24),(25) writes under the conservative form:

$$\partial_t U + \partial_x F(U) = G(x, U) \tag{26}$$

where the unknown state is $U = (A, Q)^t$.
The flux vector is $F(x, U) = (Q, \dfrac{Q^2}{A} + c^2 A)^t$ and the source term writes $G(x, U) = (0, -gA(\partial_x Z + S_f))^t$.

In the sequel, we will again suppose that the friction term vanishes.

Theorem 8 *The system (26) is strictly hyperbolic. It admits a mathematical entropy:*

$$E(A, Q, Z) = \frac{Q^2}{2A} + gAZ + c^2 A \ln A \tag{27}$$

which satisfies the entropy inequality:

$$\partial_t E + \partial_x [u(E + c^2 \ln A)] \leq 0 \quad .$$

Proof of theorem 8 Since $A(x, t) > 0$, the proof remains the same as the proof of Theorem 1.

■

Also, the system (26) admits a family of smooth steady states characterized by the relations:

$$Q = Au = C_1 \ , \tag{28}$$

$$\frac{u^2}{2} + g\, Z + c^2 \ln A = C_2 \ , \tag{29}$$

where C_1 and C_2 are two arbitrary constants. The quantity $\frac{u^2}{2}+g\,Z+c^2\ln A$ is also called the total head. Indeed, an easy computation leads to the following partial differential equation for the velocity u:

$$\partial_t u + \partial_x \left(\frac{u^2}{2} + g\,Z + c^2 \ln A \right) = 0 \quad . \tag{30}$$

Eqs. (28), (29) are thus obtained in setting the time derivative to zero in Eqs. (22), (23) and (30).

3.1 The kinetic approach

We follow the ideas used to describe the kinetic formulation for Saint-Venant equations developped above, to obtain a kinetic formulation for pressurised flow. Let us consider as before a smooth real function χ which has the following properties:

$$\chi(\omega) = \chi(-\omega) \geq 0 \, , \; \int_{\mathbb{R}} \chi(\omega) d\omega = 1, \int_{\mathbb{R}} \omega^2 \chi(\omega) d\omega = 1 \, .$$

We then define the density of particles $\mathcal{M}_{PF}(t, x, \xi)$ (*subscript PF for pressurised flow*) by the so-called *Gibbs equilibrium*:

$$\mathcal{M}_{PF}(t, x, \xi) = \mathcal{M}_{PF}(A, \xi - u) = \frac{A}{c} \chi \left(\frac{\xi - u(t, x)}{c} \right) \quad .$$

These definitions allow to obtain a kinetic representation of the system (26) by the following result.

Theorem 9 *The couples of functions (A, Q) is a strong solution of the system (26) if and only if $\mathcal{M}_{PF}(A, \xi - u)$ satisfies the kinetic equation:*

$$\frac{\partial}{\partial t} \mathcal{M}_{PF} + \xi \cdot \frac{\partial}{\partial x} \mathcal{M}_{PF} - g \frac{\partial}{\partial x} Z \cdot \frac{\partial}{\partial \xi} \mathcal{M}_{PF} = K(t, x, \xi)$$

for some collision term $K(t, x, \xi)$ which satisfies for a.e. (t, x)

$$\int_{\mathbb{R}} K \, d\xi = 0 \, , \; \int_{\mathbb{R}} \xi \, K d\, \xi = 0 \quad .$$

Proof of theorem 9 The proof relies on a very obvious computation and remains the same as the proof of Theorem 2. This is a consequence of the following relations verified by the microscopic equilibrium:

$$A = \int_{\mathbb{R}} \mathcal{M}_{PF}(\xi)\, d\xi \ , \tag{31}$$

$$Q = \int_{\mathbb{R}} \xi \mathcal{M}_{PF}(\xi)\, d\xi \ , \tag{32}$$

$$\frac{Q^2}{A} + c^2 A = \int_{\mathbb{R}} \xi^2 \mathcal{M}_{PF}(\xi)\, d\xi \ . \tag{33}$$

■

As for the Saint-Venant equations, the nonlinear pressurised flow system (26) can be viewed as a simple linear equation on a nonlinear quantity $\mathcal{M}$. Thus, we characterise the function χ which defines the density of particles $\mathcal{M}(t,x,\xi)$ in its kinetic approach, with the same interpretation as for free surface flows in term of *Gibbs equilibrium*.

Theorem 10 *Let $A(x,t)$ and $Q(x,t)$ be two given functions.*

(1) The minimum of the energy:

$$\mathcal{E}(f) = \int_{\mathbb{R}} \left(\frac{\xi^2}{2} f(\xi) + c^2 f \ln(f) + gZf(\xi) + c^2 \ln(c\sqrt{2\pi}) f(\xi) \right) d\xi \quad ,$$

under the constraints:

$$f \geq 0 \, , \ \int_{\mathbb{R}} f(\xi) d\xi = A \, , \ \int_{\mathbb{R}} \xi f(\xi) d\xi = Q \quad ,$$

is attained by the function:

$$\mathcal{M}_{PF}(t,x,\xi) = \mathcal{M}_{PF}(A, \xi - u) = \frac{A}{c} \chi \left(\frac{\xi - u(t,x)}{c} \right)$$

where χ is defined by:

$$\chi(\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\omega^2}{2} \right) \quad . \tag{34}$$

(2) Moreover, the function χ defined by (34) ensures us to have the relation

$$\mathcal{E}(\mathcal{M}_{PF}) = E(A, Q, Z)$$

if A and Q are solution of the pressurised flow equations (26) and the entropy E is defined by (27).

Proof of theorem 10

(1) The Euler-Lagrange equation associated to the minisation problem reads: $\frac{\xi^2}{2} + c^2 \ln(f) + c^2 + gZ + c^2 \ln(c\sqrt{2\pi}) = \lambda + \mu\xi$ where $\lambda(A,Q)$ and $\mu(A,Q)$ are the Lagrange multipliers. One may easily verify that the function

$f = \mathcal{M}_{PF}(A, \xi - u)$ is a solution of the minimisation problem. Moreover, as $f \geq 0$, the function $\mathcal{E}(f)$ is strictly convex which ensures as the unicity of the minimum.

(2) Writing $\mathcal{E}(\mathcal{M}_{PF}(A, \xi - u))$ and using the macroscopic representation (31), (32), (33) leads to the above equality. ■

Remark 11 It is the second point of the above theorem that motivates the choice of the function χ and the constant in the energy $\mathcal{E}$.

We conclude this kinetic formulation of the pressurised flow equations by examining if the definition (34) of the function χ ensures that the Gibbs equilibrium $\mathcal{M}_{PF}$ is solution of the still water steady state, says $u = \dfrac{Q}{A} = 0$ and $c^2 \ln M + gZ = constant$. The kinetic equation of the still water steady state writes again:

$$\xi \cdot \frac{\partial \mathcal{M}_{PF}}{\partial x} - g \frac{\partial Z}{\partial x} \cdot \frac{\partial \mathcal{M}_{PF}}{\partial \xi} = 0 \quad . \tag{35}$$

Proposition 12 *The Gibbs equilibrium $\mathcal{M}_{PF}$ satisfies the still water steady state equation (35).*

The proof of this result relies on simple computations.

4 Dual model

The two preceding models, for the free-surface flows (1)-(2) and for the pressurised flows (24),(25), are written under a conservative form and are formally very closed. The main difference arises from the pressure laws (4) and (21). The dual model thus writes:

$$\partial_t A + \partial_x Q = 0 \tag{36}$$

$$\partial_t Q + \partial_x \left(\frac{Q^2}{A} + p(x, A, E) \right) = -g\, A\, (\partial_x Z + S_f) \tag{37}$$

where E denotes the "state" of the current point x (free surface : $E = FS$, or pressurise : $E = PF$) and where the pressure law term writes:

$$\begin{cases} p(x, A, E) = g\, I_1(A) \text{ if } A \leq A_{max} \text{ and } E = FS, \\ \\ p(x, A, E) = g\, I_1(A_{max}) + c^2\, (A - A_{max}) \text{ if } E = PF, \end{cases} \tag{38}$$

and the friction term is given by the Manning-Strickler law (3) with:

$$\begin{cases} K(A,E) = \dfrac{1}{K_s^2 R_h(A)^{\frac{4}{3}}} & \text{if } A \leq A_{max} \text{ and } E = FS, \\ \\ K(A,E) = \dfrac{1}{K_s^2 R_h(A_{max})^{\frac{4}{3}}} & \text{if } E = PF. \end{cases} \tag{39}$$

Thus the dual model writes in the conservative form:

$$\partial_t U + \partial_x F(x,U) = G(x,U) \tag{40}$$

where the unknown state is $U = \begin{pmatrix} A \\ Q \end{pmatrix}$.

The flux vector is $F(x,U) = \begin{pmatrix} Q \\ \dfrac{Q^2}{A} + p(x,A,E) \end{pmatrix}$ and the source term writes $G(x,U) = \begin{pmatrix} 0 \\ -g\,A\,(\partial_x Z + S_f) \end{pmatrix}$.

Notice that, theoretically, the state of the flow (free surface or pressurised) and the position of the transition points between these two types of flow are also unknowns. The pressure defined by (38) and the friction term defined by (39) are continuous at each transition point but not the gradient of pressure. This particular fact is carefully treated numerically in [4].

At each point x of the pipe, and at each time t, if we know the state of the flow (free surface of pressurised), we are able to use a kinetic formulation according to the one presented in this article for each type of flows as follows. We define the Gibbs equilibrium by:

$$\mathcal{M}(t,x,E,\xi) = \begin{cases} \dfrac{A}{c(A)}\chi\left(\dfrac{\xi - u(t,x)}{c(A)}\right) & \text{with } c(A) = \sqrt{g\overline{y}} \quad \text{if } E = FS, \\ \dfrac{A}{c}\chi\left(\dfrac{\xi - u(t,x)}{c}\right) & \text{with } c = \sqrt{\dfrac{1}{\beta\,\rho_0}} \quad \text{if } E = PF \end{cases} \tag{41}$$

where the function χ is defined by relation (17) or (34) depending on the state of the flows. Notice that $c(A)$ is not the speed of sound for the free surface flow whereas c is the speed of sound for the pressurised one. The preceding results on the minimisation of the energy and the preservation of the still water steady state stay true. This feature is the key of the construction of a numerical kinetic scheme.

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