An implicit finite volume scheme for unsteady flows in deformable pipe-lines

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ABSTRACT. *We present a model for compressible flows in a deformable pipe which is an alternative to the Allievi equations. The numerical simulation is performed using a first order linearly implicit scheme. In the case of waterhammer we compare the numerical results with those of an extension to second order implicit scheme*

KEYWORDS: *implicit schemes, Roe scheme, boundary conditions, waterhammer*

1. Introduction

The commonly used model to describe flows in pipe-lines is the Allievi equations. These equations are usually solved with the characteristics method. The resulting system of 1st order partial differential equations cannot be written under a conservative form since this model is derived by neglecting some acceleration terms. In this paper, we derive a model from 3D compressible Euler equations by integration over sections orthogonal to the flow direction and by using a linearized pressure law $p = \dfrac{\rho - \rho_0}{\beta \rho_0}$ in which ρ represents the density of the liquid, ρ_0, the density at atmospheric pressure and β the water compressibility ratio. This model writes:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = G(x, U) \qquad [1]$$

where $U = \begin{pmatrix} \rho A \\ \rho Q \end{pmatrix} = \begin{pmatrix} M \\ D \end{pmatrix}$, $F(U) = \begin{pmatrix} D \\ \dfrac{D^2}{M} + c^2 M \end{pmatrix}$ and

$$G(x, U) = \begin{pmatrix} 0 \\ gM \sin\theta - g\, K(\delta)\, \dfrac{D|D|}{M} + \dfrac{c^2\, M}{A} \dfrac{\partial S}{\partial x} \end{pmatrix}.$$

$A = A(x,t)$ is the surface area of a section normal to the pipe axis at position x, $Q = Au$ is the flow of the liquid (with the average velocity u), g is the gravity acceleration, θ is the slope of the pipe at position x and c the sound speed. $K(\delta)$ is a positive factor depending on the diameter δ of the pipe. We use a linear elastic law for the deformation of the section, derived from the Hooke's law for an elastic material. Setting $A(x,t) = S(x, p(x,t))$, for a pipe with a circular cross section this law is, [STR 98]:

$$\frac{\partial S}{\partial p} = \frac{S\delta}{eE} \qquad [2]$$

where e is the wall thickness and E the Young's modulus of elasticity for the wall material. Then the sound speed is approximated by

$$c = \sqrt{\frac{1}{\rho_0\left(\beta + \frac{\delta}{eE}\right)}} \qquad [3]$$

instead of $\sqrt{\dfrac{1}{\rho_0 \beta}}$ in the undeformable case. The time evolution equation for A deduced from [2] writes

$$\frac{\partial A}{\partial t} = \lambda \frac{\partial M}{\partial t} \qquad [4]$$

with $\lambda = \dfrac{\frac{\delta}{e\,E}}{\beta\,\rho_0 + \rho\frac{\delta}{e\,E}} \simeq \dfrac{\delta}{e\,E}\,c^2$. It is coupled with equation [1]. Since the eigenvalues of the jacobian matrix $\mathrm{d}F(U)$ have very large magnitude (of order $10^3 \mathrm{ms}^{-1}$), we choose, following [GAL 96, EYM 00], to discretise this system of 1st order conservative partial differential equations by a linearly implicit finite volume scheme to avoid the usual CFL condition for an explicit 1st order spatial discretization. The main difficulty comes from the boundary condition treatment. We propose an adaptation of a classical method ([DUB 01, KUM 93] for instance) to the implicit case.

2. Model

This model is derived from 3D system of compressible Euler equations by integration over sections orthogonal to the flow axis . The equation for conservation of mass and the first equation for the conservation of momentum are

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\,\vec{\mathrm{U}}) = 0 \qquad [5]$$

$$\frac{\partial(\rho\,u)}{\partial t} + \mathrm{div}(\rho\,\mathrm{u}\,\vec{\mathrm{U}}) = F_x - \frac{\partial P}{\partial x} \qquad [6]$$

with the speed vector $\vec{U} = u\vec{i} + v\vec{j} + w\vec{k} = u\vec{i} + \vec{V}$. We use the linearized pressure law $P = P_a + \frac{1}{\beta}\left(\frac{\rho}{\rho_0} - 1\right)$, exterior strengths are the gravity $\vec{g}$ and the friction $-S_f\,\vec{i}$. Then equations [5]-[6] become

$$\begin{aligned} \partial_t \rho + \partial_x(\rho\, u) + \mathrm{div}_{(\mathrm{y,z})}(\rho\,\vec{\mathrm{V}}) &= 0 & [7] \\ \partial_t(\rho\, u) + \partial_x(\rho\, u^2) + \mathrm{div}_{(\mathrm{y,z})}(\rho\,\mathrm{u}\,\vec{\mathrm{V}}) &= \rho g(\sin\theta - S_f) - \frac{\partial_x \rho}{\beta\rho_0} & [8] \end{aligned}$$

Equations [7]-[8] are integrated over a cross section $\Omega(x,t)$. In the following, overlined letters represent averaged quantities over $\Omega(x,t)$. For the first equation we have successively, with the approximation $\bar{\rho u} \simeq \bar{\rho}\,\bar{u}$:

$$\begin{aligned} \int_{\Omega(x,t)} \partial_t \rho &= \partial_t \int_{\Omega(x,t)} \rho - \int_{\partial\Omega(x,t)} \rho \frac{\partial \vec{M}}{\partial t} \cdot \vec{n} \\ \int_{\Omega(x,t)} \partial_x(\rho u) &= \partial_x(\bar{\rho}\,\bar{u} A) - \int_{\partial\Omega(x,t)} \rho u \frac{\partial \vec{M}}{\partial x} \cdot \vec{n} \\ \int_{\Omega(x,t)} \mathrm{div}_{(\mathrm{y,z})}(\rho\vec{\mathrm{V}}) &= \int_{\partial\Omega(x,t)} \rho \vec{V} \cdot \vec{n} \end{aligned}$$

The water proof condition writes $\left(\frac{\partial \vec{M}}{\partial t} + u\frac{\partial \vec{M}}{\partial x} - \vec{V}\right) \cdot \vec{n} = 0$, then we get the following equation for the conservation of the mass :

$$\partial_t(\bar{\rho}A) + \partial_x(\bar{\rho}Q) = 0 \qquad [9]$$

where we have set $Q = A\,\bar{u}$. Next, with the approximations $\bar{\rho u} \simeq \bar{\rho}\,\bar{u}$ and $\bar{\rho u^2} \simeq \bar{\rho}\,\bar{u}^2$, the same procedure applied to the momentum equation [8] leads to

$$\partial_t(\bar{\rho}Q) + \partial_x\left(\bar{\rho}\frac{Q^2}{A} + \frac{\bar{\rho}}{\beta\rho_0}A\right) = g\bar{\rho}A(\sin\theta - S_f) + \frac{\bar{\rho}}{\beta\rho_0}\frac{\partial A}{\partial x} \qquad [10]$$

From [2] we get $\frac{\partial A}{\partial x} = \frac{\partial S}{\partial x} + \frac{\delta}{e\,E}A\frac{\partial p}{\partial x} = \frac{\partial S}{\partial x} + \frac{\delta}{e\,E}\frac{A}{\beta\rho_0}\frac{\partial \rho}{\partial x}$. Joined to [10], with S_f given by the Manning-Strickler law (see [STR 98]), this gives [1]-[3]. $\frac{\partial S}{\partial x}$ is related to the geometry of the pipe. For the numerical approximation of [1]- [4] we use a finite volume method with Roe's numerical flux in a partial linearly implicit version. We present here the implicit first order scheme. A spatial second order extension will be presented in a forthcoming paper.

3. The finite volume discretisation and its implicit formulation

The main axis of the pipe, with length L, is divided in N meshes $m_i = [x_{i-1/2}, x_{i+1/2}]$, $1 \leq i \leq N$ such that $x_{\frac{1}{2}} = 0$ and $x_{N+\frac{1}{2}} = L$. We denote x_i the center of m_i and h_i its length. Δt denotes the timestep. We set $t_0 = 0$ and for $n \geq 0$, $t_{n+1} = t_n + \Delta t$. The discrete unknowns are $U_i^n = \begin{pmatrix} M_i^n \\ D_i^n \end{pmatrix}$ $1 \leq i \leq N, \quad 0 \leq n \leq n_{max}$. The upstream and downstream boundary states U_0^n, U_{N+1}^n are associated to fictive meshes denoted 0 et $N+1$. Let $M \in \mathcal{M}_n(\mathbb{R})$ be a diagonalizable real matrix with real eigenvalues $\lambda_1, \cdots \lambda_n$ associated to eigenvectors $r_1, \cdots, r_n$: $\mathrm{diag}(\lambda_i) = P^{-1}\, M\, P$. We set $|M| = P\, \mathrm{diag}(|\lambda_i|)\, P^{-1}$.

3.1. *Principle of explicit first order Roe scheme*

In this section we recall the principle of the explicit first order Roe scheme applied to the system [1]- [2] without taking any boundary data into account. Roe's scheme, derived from Godunov's method, is based on the use of an approximate Riemann solver (see [EYM 00] for instance). It takes the conservative form

$$h_i \frac{U_i^{n+1} - U_i^n}{\Delta t} + F_{i+1/2}^n - F_{i-1/2}^n = h_i\, G_i^n \quad i \in \mathbb{Z}, \quad n \geq 0 \qquad [11]$$

where for all $i \in \mathbb{Z}$, U_i^0 is the averaged value of U_0 on mesh i. The numerical flux $F_{i+1/2}^n$, in its centered form, is given by

$$\Phi(U_g, U_d) = \frac{F(U_g) + F(U_d)}{2} + \frac{1}{2}|A(U_g, U_d)| \cdot (U_g - U_d) \qquad [12]$$

The timestep (at time t_n) in the resulting scheme is classically subject to a "Courant-Friedrich-Lévy type" condition:

$$(CFL) \qquad \Delta t = C \frac{\inf_{i \in \mathbb{Z}} h_i}{\max\left\{ \left| \lambda t_{k,i+1/2}^n \right| ;\ 1 \leq k \leq 2,\ i \in \mathbb{Z} \right\}} \qquad C \in]0, 1[$$

Following Roe's method ([ROE 81] in the case of unidimensional Euler equations) we get, at least in the case of a rigid pipe, the following Roe matrix

$$A(U_g, U_d) = \begin{pmatrix} 0 & 1 \\ c^2 - \tilde{u}^2 & 2\tilde{u} \end{pmatrix} \qquad [13]$$

with

$$\tilde{u} = \frac{u_g \sqrt{M_g} + u_d \sqrt{M_d}}{\sqrt{M_g} + \sqrt{M_d}} \qquad c = \sqrt{\frac{1}{\beta \rho_0}}.$$

In the deformable case we use [13] with a suitable averaged sound speed (based on [3]) ensuring the consistency of [12]. The resulting matrix is no longer a

Roe matrix but the conservativity is naturally ensured by the Finite Volume scheme [11]. Practically, with $\beta = 5.10^{-10}\, m^3/N$ the (CFL) condition gives $\Delta t \leq 0.7\, 10^{-3}\, \Delta x$. This very severe restriction motivates the construction of an implicit scheme.

3.2. *First order implicit Roe scheme*

A fully implicit scheme based on the previous approach would write

$$h_i \, \frac{U_i^{n+1} - U_i^n}{\Delta t} + F_{i+1/2}^{n+1} - F_{i-1/2}^{n+1} = h_i \, G_i^{n+1}$$

with

$$\begin{aligned} F_{i+1/2}^{n+1} &= \frac{F(U_i^{n+1}) + F(U_{i+1}^{n+1})}{2} + \frac{1}{2} \, |A(U_i^{n+1}, U_{i+1}^{n+1})| \cdot (U_i^{n+1} - U_{i+1}^{n+1}) \\ G_i^{n+1} &= G(x_i, U_i^{n+1}). \end{aligned}$$

It would obviously be of high cost and not easily extended to second order. The current approach, following [GAL 96], consists in linearizing $F(U_i^{n+1})$ and $G(x_i, U_i^{n+1})$ around U_i^n and using the explicit matrix $A(U_i^n, U_{i+1}^n)$ rather than $A(U_i^{n+1}, U_{i+1}^{n+1})$. Then we get

$$h_i \, \frac{U_i^{n+1} - U_i^n}{\Delta t} + \tilde{F}_{i+1/2}^{n+1} - \tilde{F}_{i-1/2}^{n+1} = h_i \, \tilde{G}_i^{n+1} \qquad [14]$$

In this scheme the numerical flux is given by

$$\begin{aligned} \tilde{F}_{i+1/2}^{n+1} &= \frac{1}{2} \left(\mathrm{d}F(U_i^n) \cdot U_i^{n+1} + \mathrm{d}F(U_{i+1}^n) \cdot U_{i+1}^{n+1} \right) \\ &+ \frac{1}{2} \, |A(U_i^n, U_{i+1}^n)| \cdot (U_i^{n+1} - U_{i+1}^{n+1}) \end{aligned} \qquad [15]$$

The non- zero component of the right hand side is

$$\begin{aligned} \left(\tilde{G}_i^{n+1} \right)_2 &= g M_i^{n+1} \sin\theta_i - 2g \, K_i \, \frac{|D_i^n| \, D_i^{n+1}}{M_i^n} + g \, K_i \, \frac{|D_i^n| \, D_i^n}{(M_i^n)^2} M_i^{n+1} \\ &+ \frac{M_i^{n+1}}{\beta \rho_0 A_i} \left(\frac{\partial S}{\partial x} \right)_i \end{aligned} \qquad [16]$$

From a mathematical point of view, we must give as much scalar boundary conditions as incoming characteristic curves, that is one at each end of the pipe, thus U_0^{n+1} and U_{N+1}^{n+1} are not completely specified and a special boundary conditions treatment is performed.

3.3. *Boundary conditions*

In order to achieve the description of the implicit scheme, we precise now a way to take the boundary conditions into account. We recall that the upstream and downstream state vectors (corresponding to $x_{1/2}$ and $x_{N+1/2}$) at time t_n are respectively denoted $U_0^n = \begin{pmatrix} M_0^n \\ D_0^n \end{pmatrix}$ *et* $U_{N+1}^n = \begin{pmatrix} M_{N+1}^n \\ D_{N+1}^n \end{pmatrix}$. The method that we describe below is closely related to those studied by Dubois [DUB 01], Kumbaro [KUM 93] (see also [EYM 00]). It allows the computation of the boundary states using known values at the same time, so it is naturally implicit. Applying this method for nonlinear boundary conditions (BC in brief) at time t_{n+1} is not compatible with the framework of linearized implicitation as above. We propose to apply it at time t_{n+1} after a first step which consists in completing [14]-[15]-[16] with two linear equations thanks to a modified procedure that we precise further. Solving the resulting system supplies a first estimate of interior and unknown boundary states, then we make use of the standard BC method. Following [DUB 01, EYM 00, KUM 93] we start with given interior vector states U_i^n $(1 \leq i \leq N)$ and one component of each boundary vector states U_0^n and U_{N+1}^n or any relationship between those components. We have to build complete boundary states using these data at the same time and not at the previous one as in the characteristic method. Let us consider the upstream state for instance. The Roe matrix $A = A(U_0^n, U_1^n)$ has two eigenvalues $\tilde{\lambda}_1 < 0 < \tilde{\lambda}_2$ joined with two eigenvectors $\tilde{r}_1$ et $\tilde{r}_2$ (depending on the unknown part of U_0^n). The vector states U_0^n and U_1^n are expressed in the basis of eigenvectors: $U_1^n = \alpha_1^n \tilde{r}_1 + \alpha_2^n \tilde{r}_2$ and $U_0^n = \alpha_1 \tilde{r}_1 + \alpha_2 \tilde{r}_2$. The BC method consists in setting $\alpha_1 = \alpha_1^n$. Then we get U_0^n solving the a priori nonlinear system

$$\begin{cases} \alpha_1 = \alpha_1^n \\ \text{Boundary datum.} \end{cases} \qquad [17]$$

Our adaptation to the implicit scheme consists in two steps.

First step : we apply the previous procedure using the eigenvectors of the Roe matrix $A(U_0^n, U_1^n)$ at time t_n instead of time t_{n+1}. From [17] we get: $D_0^{n+1} = D_1^{n+1} - (\tilde{u}_{1/2}^n + c)(M_1^{n+1} - M_0^{n+1})$. Similarly for downstream state we get: $M_{N+1}^{n+1} = M_N^{n+1} + \dfrac{D_N^{n+1} - D_{N+1}^{n+1}}{c - \tilde{u}_{N+1/2}^n}$. The linear system arising from [14] is now completely determined.

Second step : equipped with interior vector states issued from the first step we apply the standard BC method at time t_{n+1}. This leads to:

$$D_0^{n+1} = D_1^{n+1} \frac{M_0^{n+1}}{M_1^{n+1}} + c\,(M_0^{n+1} - M_1^{n+1}) \sqrt{\frac{M_0^{n+1}}{M_1^{n+1}}}$$

$$c\,(M_{N+1}^{n+1} - M_N^{n+1}) \sqrt{M_{N+1}^{n+1} M_N^{n+1}} + D_{N+1}^{n+1} M_N^{n+1} - D_N^{n+1} M_{N+1}^{n+1} = 0.$$

Commonly used boundary conditions are for instance an upstream constant total load and the downstream flow (as in the presented numerical results).

The total load in the duct is defined by $H = z + \dfrac{u^2}{2g} + p$, where z is the altitude (m), u the flow speed (m/s), p the relative pressure (m) defined by $p = \dfrac{\rho - \rho_0}{\beta \rho_0^2 g}$. We impose at the entrance of the duct with altitude z_0 a constant load H_0. Thus we get $\left(\dfrac{D_0^{n+1}}{M_0^{n+1}}\right)^2 + \dfrac{2M_0^{n+1}}{\beta \rho_0^2 A_0} = n2g(H_0 - z_0) + \dfrac{2}{\beta \rho_0}$ as boundary datum in [17]. A similar technique as above is then applied.

4. Numerical validation

We present now numerical results of a (severe) waterhammer test. The circular pipe of diameter 2 m and thickness 20 cm is 2000 m long. The altitude of the upstream end of the pipe is 250 m and the angle is 5°. The Young coefficient is $23.\,10^9$ Pa. The total upstream load is 300 m. The initial downstream flow is 30 m^3/s and we cut the flow in 10 seconds. The spatial mesh size is 20 m and the time step is 10^{-3} s. Figure 1 (left) represents the piezometric line ($z+p$ (m)) at the middle of the pipe considered as undeformable or deformable whereas Figure 1 (right) represents the variation of the diameter. One can see that in the deformable case, the pressure is less high since the pipe absorbs a great part of the constraint. One can remark that the first order scheme is very diffusive. To reduce the diffusive effect of the finite volume discretisation, we performed a MUSCL like second order implicit scheme that we will present in a forthcoming paper. Figure 2 represents the comparaison between the 1st order and the second order scheme by plotting the piezometric line at the middle of the undeformable pipe.

References

[DUB 01] DUBOIS F., "An introduction to finite volumes", for publication in: *Comp. Methods and Algorithms*, O. Pironneau and V. Shaidurov eds.

[EYM 00] EYMARD R., GALLOUET T. AND HERBIN R., "Finite volume methods", *Handbook of Numerical Analysis* , P. Ciarlet et J.L. Lions eds., North Holland, 2000.

[GAL 96] GALLOUET T., "Rough schemes for complex hyperbolic systems", in: *Finite Volumes For Complex Applications*, Hermes, 1996.

[KUM 93] KUMBARO A., "Modélisation, analyse mathématique et numérique des modèles bi-fluides d'écoulements diphasiques", Thèse de l'Université Paris XI, 1993.

[ROE 81] ROE P.L., "Approximate Riemann Solvers, Parameter Vectors and Difference Schemes", *J. of Comp. Ph.*, vol. 43, 1981, pp. 357-372.

[STR 98] STREETER V.L., WYLIE E.B. AND BEDFORD K.W. *Fluid Mechanics*, McGraw-Hill, 1998

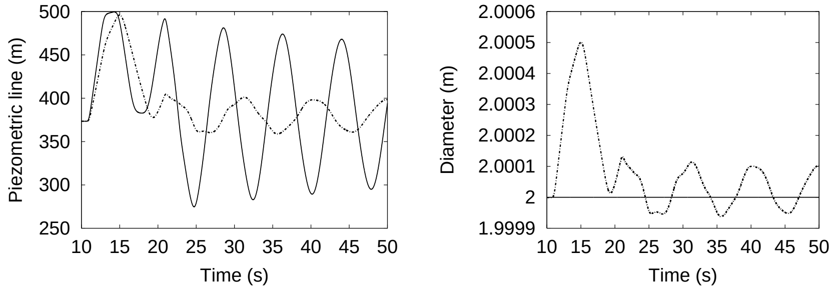

Figure 1. *Influence of deformation of the pipe: piezometric line (left) and diameter (right) at the middle of the pipe (Solid line represents the undeformable pipe, dashed line the deformable pipe).*

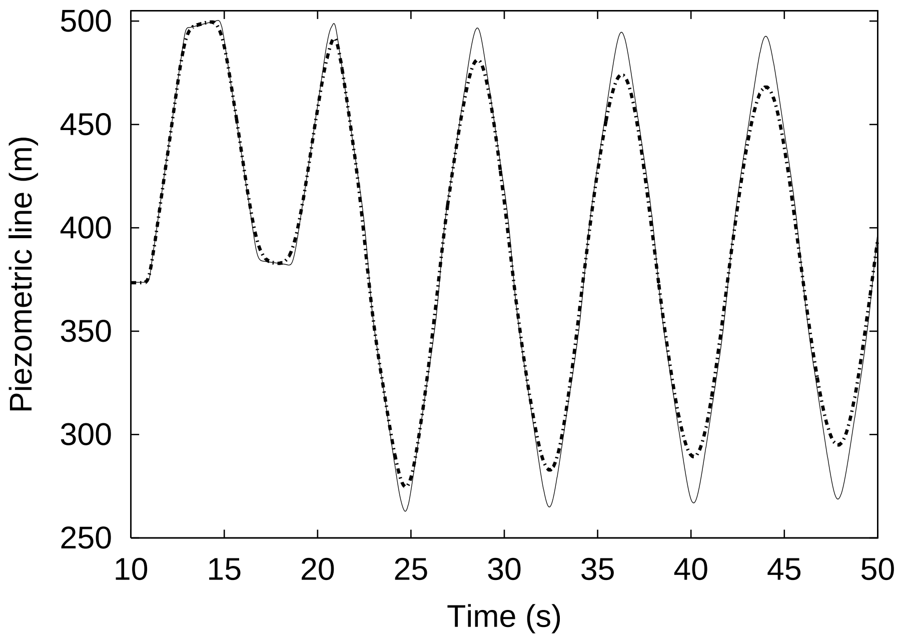

Figure 2. *Comparaison of the first order and second order scheme: piezometric line at the middle of the undeformable pipe (Solid line represents the second order scheme, dashed line the first order scheme).*