A numerical scheme for the propagation of internal waves in an oceanographic model.

Christian Bourdarias, Stéphane Gerbi and Ralph Lteif

Abstract In this paper, we introduce a new reformulation of the Green-Naghdi model in the Camassa-Holm regime for the propagation of internal waves over a flat topography to improve the frequency dispersion of the original model. We develop a second order splitting scheme where the hyperbolic part of the system is treated with a high-order finite volume scheme and the dispersive part is treated with a finite difference approach. Numerical simulations are then performed to validate the model.

1 Introduction

This study deals with the propagation of internal waves in the uni-dimensional setting located at the interface between two layers of fluids of different densities. The fluids are assumed to be incompressible, homogeneous, and immiscible, limited from above by a rigid lid and from below by a flat bottom. This type of fluid dynamics problem is encountered by researchers in oceanography when they study the wave near the shore. Because of the difference in the salinity of the different layers of water near the shore, it is useful to model the flow of salted water by a two layers

Christian Bourdarias
LAMA, UMR 5127 CNRS, Université Savoie Mont Blanc,
73376 Le Bourget du lac cedex, France
e-mail: christian.bourdarias@univ-smb.fr

Stéphane Gerbi
LAMA, UMR 5127 CNRS, Université Savoie Mont Blanc,
73376 Le Bourget du lac cedex, France
e-mail: stephane.gerbi@univ-smb.fr

Ralph Lteif
Laboratory of Mathematics-EDST,
Lebanese University, Beirut, Lebanon
e-mail: ralphlteif_90@hotmail.com

incompressible fluids flow. The usual way of describing such a flow is to use the 3D-Euler equations for the different layers adding some thermodynamic and dynamic conditions at the interface. This system will be called the *full Euler system*.

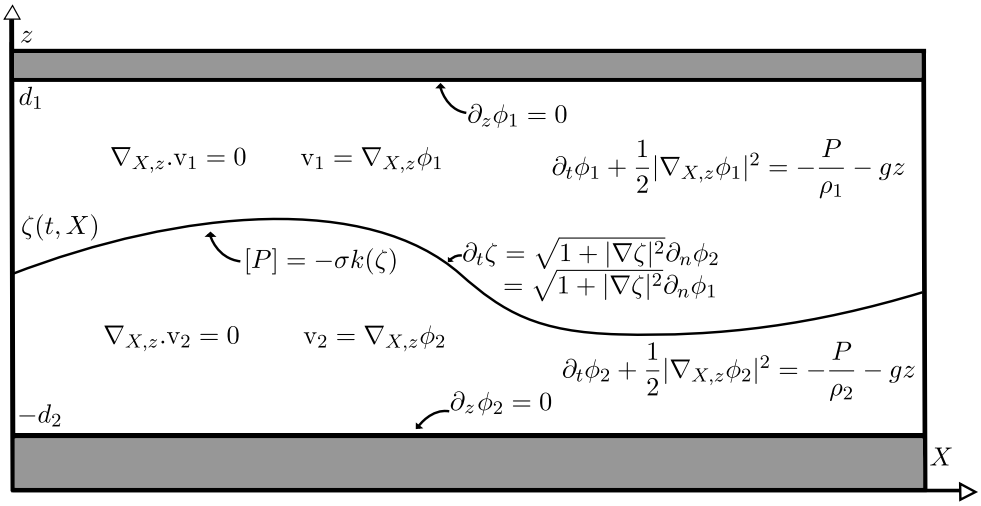

Fig. 1 Domain of study and governing equations.

We introduce dimensionless variables and the two scale parameters μ, the shallowness parameter and ε, the nonlinearity parameter, defined by: $\mu = \frac{d_1^2}{\lambda^2}$, $\varepsilon = \frac{a}{d_1}$ where a is a typical length of the vertical oscillation of the interface, λ is a typical wavelength. We also define the dimensionless parameters $\gamma = \frac{\rho_1}{\rho_2}$ and $\delta = \frac{d_1}{d_2}$ representing respectively the ratio between the densities and the depth of the two layers.

In this work, we present a splitting technique for the numerical resolution of the GN model in the Camassa-Holm (or medium amplitude internal waves) regime, $\varepsilon = \mathcal{O}(\sqrt{\mu})$, obtained and fully justified by Duchêne, Israwi and Talhouk in [4]. In the Camassa-Holm regime, the authors has proved the existence and well-posedeness of the resulting system and its consistecy with the *full Euler system* in the sense that its solution remain close to the exact solution of the *full Euler system* with corresponding initial data up to the order $\mathcal{O}(\mu^2)$.

This model is first recast under a new formulation more suitable for numerical resolution with the same order of precision as the standard one but with improved frequency dispersion. For this sake, we introduce a one parameter family depending on $\alpha > 0$. The choice of the parameter α is motivated by the exact agreement between the phase velocity dispersion relation of the *full Euler system* and the improved Green-Naghdi system (1) for fixed values for γ and δ and for large wavenumbers (k around 4-5). This parameter is denoted α_{opt} and is obtained by an algebraic equation, see [2] for details.

We obtain the Green-Naghdi model in the Camassa-Holm with improved dispersion whose unknowns are ζ the mean elevation of the interface and v the shear mean velocity :

$$\begin{cases} \partial_t \zeta + \partial_x\big(f(\varepsilon\zeta)v\big) \;=\; 0, \\ \\ (I+\mu\nu\alpha T[0])\big[\partial_t v + \varepsilon\varsigma v\partial_x v + \dfrac{\alpha-1}{\alpha}\big((\gamma+\delta)\partial_x\zeta + \varepsilon\partial_x(q_3(\varepsilon\zeta)v^2)\big)\big] \\ +\dfrac{1}{\alpha}\big((\gamma+\delta)\partial_x\zeta + \varepsilon\partial_x(q_3(\varepsilon\zeta)v^2)\big) + \mu\varepsilon Q_1(v) + \mu\varepsilon\nu Q_2(\zeta) + \mu\varepsilon\nu Q_3(\zeta) = 0. \end{cases} \tag{1}$$

with $f(X)=\frac{(1-X)(\delta^{-1}+X)}{1-X+\gamma(\delta^{-1}+X)}$, $T[0]V=-\partial_x^2 V$, $S[\zeta]V=-\kappa_2\partial_x(\zeta\partial_x V)$, $q_3(\varepsilon\zeta)=\frac{1}{2}\big(f'(\varepsilon\zeta)-\varsigma\big)$,
$Q_1(v)=\kappa\partial_x((\partial_x v)^2)$, $Q_2(\zeta)=-S[\zeta](I+\mu\nu\alpha T[0])^{-1}\big((\gamma+\delta)\partial_x\zeta\big)$,
$Q_3(\zeta)=\kappa_1\zeta T[0](I+\mu\nu\alpha T[0])^{-1}[(\gamma+\delta)\partial_x\zeta]$.

2 Numerical methods

As pointed out by many authors [1, 8] the improved dispersion Green-Naghdi equations (1) is well-adapted to the implementation of a splitting scheme separating the hyperbolic and the dispersive parts of the equations.

2.1 The splitting method

We decompose the solution operator $S(.)$ associated to (1) at each time step Δt by the following second order operator splitting:

$$S(\Delta t) = S_1(\Delta t/2)S_2(\Delta t)S_1(\Delta t/2)$$

where $S_1(.)$ is the solution operator associated to the conservative part, and $S_2(.)$ the solution operator associated to the dispersive part of the equations (1). In this study, S_1 is computed using a finite volume method while S_2 is computed using a classical finite-difference method.

• $S_1(t)$ is the solution operator associated to the conservative part namely the nonlinear shallow water equations, NSWE:

$$\begin{cases} \partial_t \zeta + \partial_x\big(f(\varepsilon\zeta)v\big) \;=\; 0, \\ \\ \partial_t v + \partial_x\Big(\dfrac{\varepsilon f'(\varepsilon\zeta)}{2}v^2 + (\gamma+\delta)\zeta\Big) = 0. \end{cases} \tag{2}$$

Under the hyperbolicity condition for the shallow water system provided in [6], this system is strictly hyperbolic.

• $S_2(t)$ is the solution operator associated to the remaining (dispersive) part of the equations.

$$\begin{cases} \partial_t \zeta \ = \ 0, \\ \\ (I+\mu\nu\alpha T[0])\big[\partial_t v - \dfrac{1}{\alpha}\big((\gamma+\delta)\partial_x\zeta + \varepsilon\partial_x(q_3(\varepsilon\zeta)v^2)\big)\big] \\ +\dfrac{1}{\alpha}\big((\gamma+\delta)\partial_x\zeta + \varepsilon\partial_x(q_3(\varepsilon\zeta)v^2)\big) + \mu\varepsilon Q_1(v) + \mu\varepsilon\nu Q_2(\zeta) + \mu\varepsilon\nu Q_3(\zeta) = 0. \end{cases} \tag{3}$$

2.2 Finite volume scheme

In what follows, we consider the numerical approximation of the hyperbolic system of conservation laws (2). We have constructed three finite volume schemes: first order, second order "MUSCL" type method and finally 5th order WENO method and tested their accuracy by using the exact (up to the order $\mathcal{O}(\mu^2)$) solitary wave solutions of the one layer Green-Naghdi equations over a flat bottom (see [8]). We do not present the results in this paper but the 5th order method is clearly much more accurate. However we do not obtain the predicted order with respect with the spatial mesh size. This might be due to the fact that the given analytic solution satisfies the model up to an $\mathcal{O}(\mu^2)$ remainder. We believe that this splitting strategy may be also applied in the variable bottom case. This is the subject of a future work.

2.2.1 Higher order finite-volume scheme: WENO5-RK4

To reach higher order accuracy in smooth regions and a good resolution around discontinuities, we implement fifth-order accuracy WENO reconstruction, following [7]. To automatically achieve high order accuracy and non-oscillatory property near discontinuities, WENO schemes use the idea of adaptive stencils in the reconstruction procedure based on the local smoothness of the numerical solution.

As far as time discretization is concerned, we use the fourth-order explicit RungeKutta "RK4" method.

2.3 Finite difference scheme for the dispersive part

The finite volume-finite difference mix imply to switch between the cell-averaged and nodal values for each unknown and at each time step. To this end, we use the fifth-order accuracy WENO reconstruction, that allows to approximate the nodal values (i.e finite difference unknowns) $(U_i^n)_{i=1,N+1}$ in terms of the cell-averaged values (i.e finite volume unknowns) $(\overline{U}_i^n)_{i=1,N}$.

The finite difference discretization of the system (3) leads to the following discrete problem:

$$\begin{cases} \dfrac{\zeta^{n+1}-\zeta^n}{\Delta t} = 0, \\ \dfrac{v^{n+1}-v^n}{\Delta t} - \dfrac{1}{\alpha}(\gamma+\delta)D_1(\zeta^n) - 2\dfrac{\varepsilon}{\alpha}q_3(\varepsilon\zeta^n)v^nD_1(v^n) - \dfrac{\varepsilon^2}{\alpha}q_3'(\varepsilon\zeta^n)D_1(\zeta^n)(v^n)(v^n) \\ +(I-\mu\nu\alpha D_2)^{-1}\Big[\dfrac{1}{\alpha}(\gamma+\delta)D_1(\zeta^n)+2\dfrac{\varepsilon}{\alpha}q_3(\varepsilon\zeta^n)v^nD_1(v^n)+\dfrac{\varepsilon^2}{\alpha}q_3'(\varepsilon\zeta^n)D_1(\zeta^n)(v^n)(v^n) \\ +\mu\varepsilon Q_1(v^n)+\mu\nu\varepsilon Q_2(\zeta^n)+\mu\varepsilon\nu Q_3(\zeta^n)\Big] = 0, \end{cases} \tag{4}$$

with

$$Q_1(v^n) = 2\kappa D_1(v^n)D_2(v^n),$$

$$Q_2(\zeta^n) = \kappa_2 D_1\Big[\zeta^n D_1\Big((I-\mu\nu\alpha D_2)^{-1}(\gamma+\delta)D_1(\zeta^n)\Big)\Big],$$

$$Q_3(\zeta^n) = -\kappa_1\zeta^n D_2\Big[(I-\mu\nu\alpha D_2)^{-1}(\gamma+\delta)D_1(\zeta^n)\Big].$$

The system (4) is solved at each time step using a classical finite-difference technique, where the matrices D_1 and D_2 are the classical centered discretizations of the derivatives ∂_x and ∂_x^2 given below:

$$(\partial_x U)_i = \frac{1}{12\Delta x}(-U_{i+2}+8U_{i+1}-8U_{i-1}+U_{i-2}),$$

$$(\partial_x^2 U)_i = \frac{1}{12\Delta x^2}(-U_{i+2}+16U_{i+1}-30U_i+16U_{i-1}-U_{i-2}).$$

For time discretization, the fourth-order formula "DF4" is associated to a fourth-order classical Runge-Kutta "RK4" scheme, and thus one obtains the "DF4-RK4" scheme.

We only treat either periodic boundary conditions or reflective boundary conditions for the hyperbolic and dispersive parts of the splitting scheme. Suitable relations are imposed on both cell-averaged and nodal quantities.

3 Numerical validations : Kelvin-Helmholtz instabilities

In this section, we present a numerical experiment to validate the numerical efficiency and accuracy of the improved Green-Naghdi model (1). We use the WENO5 reconstruction for the hyperbolic part of the splitting scheme and a fourth order finite difference scheme "DF4" for the dispersive part, both associated to a fourth-order classical Runge-Kutta "RK4" time scheme called "WENO5-DF4-RK4". We would like to highlight the importance of the choice of the parameter α in order to improve

the frequency dispersion of the model (1), through the simulation of a sufficiently regular initial wave, following the numerical experiments performed in [5]. In the aforementioned paper they introduce a new class of Green-Naghdi type models for the propagation of internal waves with improved frequency dispersion in order to prevent high-frequency Kelvin-Helmholtz instabilities. These models are obtained by regularizing the original Green-Naghdi one by slightly modifying the dispersion components using a class of Fourier multipliers. They represent three different choices of the Fourier multipliers, each one yields to a specific Green-Naghi model which they denote as follows: **"original"** as the classical Green-Naghi model introduced in [3], **"regularized"** which is a well-posed system for sufficiently small and regular data, even in absence of surface tension, **"improved"** whose dispersion relation is the same as the one of the *full Euler system*. In order to compare with the numerical experiments done in [5], we choose the initial data $\zeta(0,x) = -e^{-4|x|^2}$ and $v(0,x) = 0$ (represented by the dashed lines). The computational domain is the interval $x \in (-4,4)$ discretized with 512 cells using periodic boundary conditions.

The dimensionless parameters are set as follows: $\mu = 0.1$, $\varepsilon = 0.5$, $\delta = 0.5$, $\gamma = 0.95$. With thess values and choosing the wavenumber $k = 5$, we obtain $\alpha_{opt} = 1.271$.

Figures 2 and 3 show the comparisons between our numerical solution for $\alpha = 1$ (left) and $\alpha_{opt} = 1.271$ (right) and the Green-Naghdi models solutions obtained in [5], with a small amount of surface tension, at time $t = 2$ and $t = 3$ respectively. We observe an excellent agreement between our numerical solution computed for $\alpha_{opt} = 1.271$ and both "improved" and "regularized" models at $t = 2$ and $t = 3$. As expected, at $t = 3$ the original model induces Kelvin-Helmholtz instabilities. Meanwhile, the flows predicted by the regularized and improved models and by our model (1) with $\alpha_{opt} = 1.271$ remain smooth and are very similar. Similarly, Figure 4 shows an excellent agreement between the numerical solutions computed for $\alpha_{opt} = 1.271$ with the "improved" and "regularized" models, without surface tension at time $t = 2$, while the flow of the original model is completely destroyed due to Kelvin-Helmholtz instabilities.

The overall observations show the importance of the choice of the parameter α in improving the frequency dispersion. Indeed, when choosing $\alpha_{opt} = 1.271$, we observe an excellent matching between our numerical solutions and those obtained by the "improved" model before the latter is completely destroyed in absence of surface tension due to the Kelvin-Helmholtz instabilities. As well, our numerical solution matches the one computed by the "regularized" model even for a large time and with or without surface tension. This is not the case when choosing $\alpha = 1$. In fact, the "improved" model has exactly the same dispersion relation as the one of the *full Euler system* and for all wave numbers (see [5, Section 3] for more details) and the dispersion relation of the "regularized" model fit the one of the *full Euler system* to an $\mathcal{O}(\mu^3)$ order. This explains the reason behind the matching when choosing an optimal value for α and highlight the advantage of the proposed approach in improving the frequency dispersion.

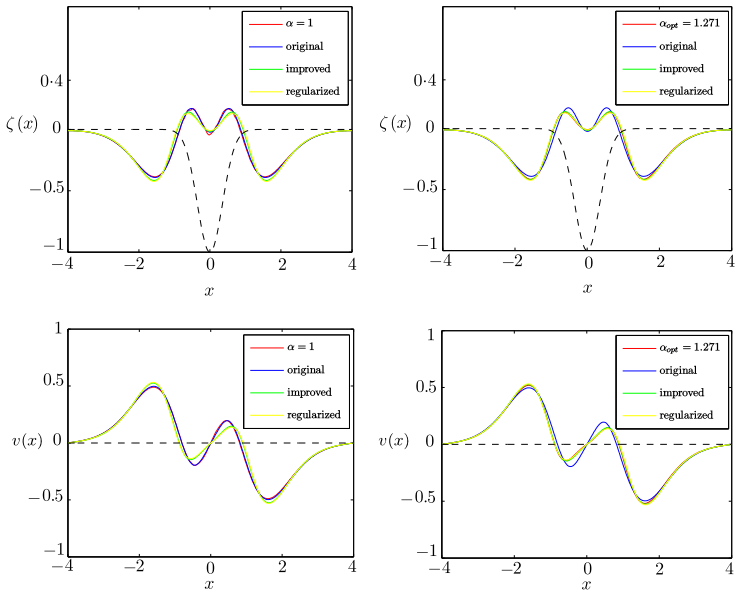

Fig. 2 Comparison with the Green-Naghdi models, with surface tension, at time $t = 2$, for $\alpha = 1$ (left) and $\alpha_{opt} = 1.271$ (right)

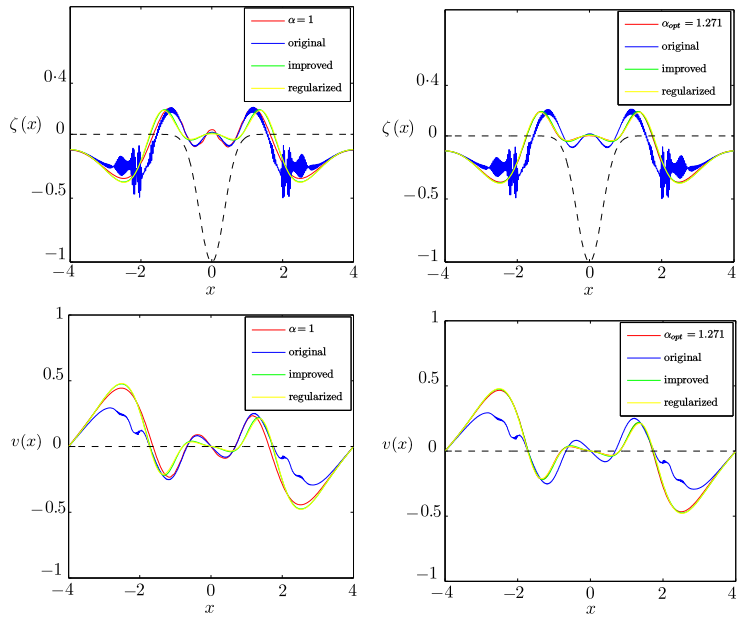

Fig. 3 Comparison with the Green-Naghdi models, with surface tension, at time $t = 3$, for $\alpha = 1$ (left) and $\alpha_{opt} = 1.271$ (right)

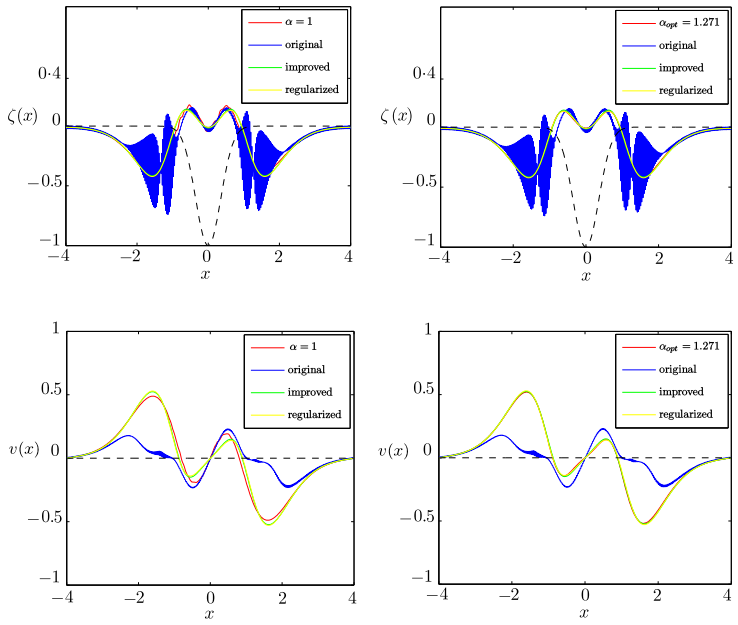

Fig. 4 Comparison with the Green-Naghdi models, without surface tension ($\mathrm{bo}^{-1} = 0$), at time $t = 2$, for $\alpha = 1$ (left) and $\alpha_{opt} = 1.271$ (right)

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