A Fractional Step Method to Simulate Mixed Flows in Pipes with a Compressible Two-Layer Model

Charles Demay, Christian Bourdarias, Benoît de Laage de Meux, Stéphane Gerbi and Jean-Marc Hérard

Abstract The so-called mixed flows in pipes include two-phase stratified regimes as well as single-phase pressurized regimes with transitions. It is proposed to handle those configurations numerically with the compressible two-layer model developed in [7]. Thus, a fractional step method is proposed to deal explicitly with the *slow* propagation phenomena and implicitly with the *fast* ones. It results in a large time-step scheme accurate in both regimes. Numerical experiments are performed including convergence results and academical test cases.

Key words: Two-layer model, implicit-explicit scheme, mixed flow

1 Introduction

We focus on air-water flows in pipes and particularly on the so-called mixed flows. The latter include stratified regimes driven by *slow* surface waves as well as pressurized regimes (pipe full of water or air) driven by *fast* acoustic waves. This type of flow occurs in piping systems of several industrial areas such as nuclear and hydraulic power plants or sewage pipelines.

Numerous modelling and numerical issues are tackled when dealing with mixed flows due to the different nature of each regime. Using a 1D approach, a model with an associated numerical scheme is proposed in [1] without computing the air phase. With the aim of accounting for air-water interactions, a compressible two-

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layer model is developed in [7]. It results in an hyperbolic two-phase two-pressure model which presents strong similarities with the isentropic form of two-fluid models introduced in [3]. In that framework, classical explicit schemes bring large numerical diffusivity in the *slow* stratified regime.

Thus, a fractional step method is derived herein to split the *slow* dynamics from the *fast* dynamics and adapt the numerical treatment. This approach is used in [2, 5] for the Baer-Nunziato model and more recently in [6] for the model under consideration. Furthermore, an implicit-explicit time discretization is also proposed in the sequel to end up with a large time-step scheme and get accuracy in the stratified regime. Contrary to the work presented in [6], the overall approach is driven by the fast pressure relaxation and the shallow-water structure of the system such that interesting results are obtained even for low speed flows.

2 The Compressible Two-Layer Model

The considered model deals with stratified gas-liquid flows in pipes. It results from a depth-averaging of the isentropic Euler set of equations for each phase where the classical hydrostatic assumption is made for the liquid, see [7] for details. Considering a two-layer air-water flow through a pipe of height H, it reads:

$$\begin{cases} \partial_t h_1 + U_I \partial_x h_1 = \lambda_p (P_I - P_2(\rho_2)), \\ \partial_t m_k + \partial_x m_k u_k = 0, \\ \partial_t m_k u_k + \partial_x m_k u_k^2 + \partial_x h_k P_k(\rho_k) - P_I \partial_x h_k = (-1)^k \lambda_u (u_1 - u_2), \end{cases} \qquad (\mathscr{S})$$

where $k = 1$ for water, $k = 2$ for air, $m_k = h_k \rho_k$ and $h_1 + h_2 = H$. Here, h_k, ρ_k, $P_k(\rho_k)$ and u_k denote respectively the height, the mean density, the mean pressure and the mean velocity of phase k. The interfacial dynamics is represented by the transport equation on h_1 while the other two equations account for mass and momentum conservation in each phase. The interfacial pressure is denoted by P_I and closed by the hydrostatic constraint, while the interfacial velocity is denoted by U_I and closed following an entropy inequality, one obtains (see [7]):

$$(U_I, P_I) = (u_2, P_1(\rho_1) - \rho_1 g \frac{h_1}{2}), \qquad (1)$$

where g is the gravity field magnitude. As the phases are compressible, state equations are required for gas and liquid pressures. For instance, perfect gas law may be used for air and linear law for water. The celerity of acoustic waves is defined by $c_k = \sqrt{P_k'(\rho_k)}$. In practice, one uses $\lambda_p = \frac{3h_1h_2}{4\pi\mu_1 H}$ and $\lambda_u = \frac{h_1h_2}{2H^2} f_i \rho_2 |u_2 - u_1|$, where μ_1 is the dynamic viscosity of water and f_i is a friction factor, see [6] for details.

Properties of $(\mathscr{S})$

(i) Smooth solutions of $(\mathscr{S})$ *comply with an entropy inequality.*

(ii) The convective part of $(\mathscr{S})$ *is hyperbolic under the condition* $|u_1 - u_2| \neq c_1$. *Its eigenvalues are unconditionally real and given by* $\lambda_1 = u_2$, $\lambda_{2,3} = u_1 \pm c_1$, $\lambda_{4,5} = u_2 \pm c_2$. *The field associated with the 1-wave is linearly degenerate while the other fields are genuinely nonlinear.*

(iii) Unique jump conditions hold within each isolated field.

(iv) The positivity of h_k *and* ρ_k *is verified.*

The details and proofs are provided in [7]. Two additional properties of $(\mathscr{S})$ are used in the proposed fractional step method. Firstly, using (1), the momentum equation for water can be written under a *Saint-Venant*-like form (see [8]):

$$\partial_t m_1 u_1 + \partial_x m_1 u_1^2 + \partial_x \rho_1 g \frac{h_1^2}{2} + h_1 \partial_x P_I = \lambda_u (u_2 - u_1). \tag{2}$$

Secondly, the pressure relaxation in the first equation of $(\mathscr{S})$ writes classically:

$$P_I \underset{t\to\infty}{\rightarrow} P_2, \tag{3}$$

and this relaxation is very fast in our framework as $\lambda_p \gg 1$. In addition, regarding the pressurized regime, $(\mathscr{S})$ degenerates towards a single-phase Euler system when one phase vanishes, as soon as the source terms also vanish.

3 Fractional Step Method Adapted to Mixed Flows

In order to handle both regimes included in mixed flows, the proposed fractional step method splits $(\mathscr{S})$ into three sub-systems. The *material* component of $(\mathscr{S})$ is treated in $(\mathscr{S}_m)$ including the pressure relaxation source term and using the Saint-Venant structure (2) for the water phase:

$$\begin{cases} \partial_t h_1 + u_2 \partial_x h_1 = \lambda_p (P_I - P_2), \\ \partial_t m_k + \partial_x m_k u_k = 0, \; k = 1,2, \\ \partial_t m_1 u_1 + \partial_x m_1 u_1^2 + \partial_x \rho_1 g \dfrac{h_1^2}{2} = 0, \\ \partial_t m_2 u_2 + \partial_x m_2 u_2^2 = 0. \end{cases} \tag{$\mathscr{S}_m$}$$

$(\mathscr{S}_a)$ refers to the *acoustic* component of $(\mathscr{S})$ including the pressure gradients:

$$\begin{cases} \partial_t h_k = 0, \; \partial_t m_k = 0, \; k = 1,2, \\ \partial_t m_1 u_1 + h_1 \partial_x P_I = 0, \\ \partial_t m_2 u_2 + h_2 \partial_x P_2 + (P_2 - P_I) \partial_x h_2 = 0, \end{cases} \tag{$\mathscr{S}_a$}$$

where $P_I = P_1(\rho_1) - \rho_1 g \frac{h_1}{2}$. Finally, $(\mathscr{S}_u)$ deals with the velocity relaxation source terms:

$$\partial_t h_k = 0, \; \partial_t m_k = 0, \; \partial_t m_k u_k = (-1)^k \lambda_u (u_1 - u_2), \; k = 1,2. \tag{$\mathscr{S}_u$}$$

A key feature is that the fast relaxation (3) solved in $(\mathscr{S}_m)$ is explicitly seen by $(\mathscr{S}_a)$.

Proposition 1 (Hyperbolicity of $(\mathscr{S}_m)$). *The convective part of $(\mathscr{S}_m)$ is weakly hyperbolic. Its eigenvalues are given by* $\{u_2; u_1 \pm \sqrt{g\frac{h_1}{2}}\}$.

$(\mathscr{S}_a)$ is not hyperbolic as its spectrum reduces to zero. This singularity is handled in the sequel using a relaxation approach.

In the discrete setting, the time step is denoted Δt and the space step Δx. The space is partitioned into cells $C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[$ where $x_{i+\frac{1}{2}} = (i+\frac{1}{2})\Delta x$ are the cell interfaces. At times $t^n = n\Delta t$, the solution is approximated on each cell C_i by $\mathbf{W}_i^n = \Big((h_1)_i^n, (h_1\rho_1)_i^n, (h_2\rho_2)_i^n, (h_1\rho_1u_1)_i^n, (h_2\rho_2u_2)_i^n\Big)^T$.

Step 1: Explicit scheme for $(\mathscr{S}_m)$. In this step, W_i is updated from W_i^n to W_i^*. A classical explicit finite-volume scheme with Rusanov fluxes is used on the convective part while the pressure relaxation source term is treated implicitly. It writes:

$$\mathbf{W}_i^* = \mathbf{W}_i^n - \frac{\Delta t}{\Delta x}\Big(\mathbf{F}(\mathbf{W}_{i+\frac{1}{2}}^n) - \mathbf{F}(\mathbf{W}_{i-\frac{1}{2}}^n)\Big) - \frac{\Delta t}{2\Delta x}\mathbf{B}(\mathbf{W}_i^n)\Big(\mathbf{W}_{i+1}^n - \mathbf{W}_{i-1}^n\Big) + \mathbf{S}(\mathbf{W}_i^*), \tag{4}$$

where $\mathbf{F}(\mathbf{W}) = (0, m_1u_1, m_2u_2, m_1u_1^2 + m_1 g\frac{h_1}{2}, m_2u_2^2)^T$, $\mathbf{B}(\mathbf{W}) = (u_2, 0, 0, 0, 0)^T$ and $\mathbf{S}(\mathbf{W}) = (\lambda_p(P_I - P_2), 0, 0, 0, 0)^T$. The fluxes are defined by:

$$\begin{cases} \mathbf{F}(\mathbf{W}_{i+\frac{1}{2}}^n) = \dfrac{1}{2}\Big(\mathbf{F}(\mathbf{W}_i^n) + \mathbf{F}(\mathbf{W}_{i+1}^n) - r_{i+\frac{1}{2}}(\mathbf{W}_{i+1}^n - \mathbf{W}_i^n)\Big), \\ r_{i+\frac{1}{2}} = \max\limits_{j\in\{i;i+1\}}\Big(|u_{2,j}^n|; \big|\big(u_1 \pm \sqrt{g\dfrac{h_1}{2}}\big)_j^n\big|\Big). \end{cases} \tag{5}$$

In order to solve implicitly the source term, the mass terms $m_{k,i}^n$ are updated first and the first equation in $(\mathscr{S}_m)$ is solved under the form $f(h_{1,i}^*) = 0$ where:

$$f(y) = y - h_{1,i}^n + \Delta t \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_2^n \frac{\partial h_1^n}{\partial x} dx - \Delta t \lambda_{p,i}^n \Big(P_I\Big(\frac{m_{1,i}^*}{y}\Big) - P_2\Big(\frac{m_{2,i}^*}{H-y}\Big)\Big). \tag{6}$$

One may easily demonstrate that f is strictly increasing on $[0;H]$ with the limits $f \underset{0^+}{\rightarrow} -\infty$ and $f \underset{H^-}{\rightarrow} +\infty$, such that $f(x) = 0$ admits a unique solution $h_{1,i}^*$ on $[0;H]$.

Proposition 2 (Positivity of heights and densities). *The proposed scheme for $(\mathscr{S}_m)$ ensures the positivity of heights and densities under the classical CFL condition:*

$$\frac{\Delta t}{\Delta x}\max_i\Big(\frac{r_{i+\frac{1}{2}} + r_{i-\frac{1}{2}}}{2}\Big) < 1, \tag{7}$$

which only implies material velocities.

Step 2: Implicit relaxation approach for $(\mathscr{S}_a)$. In this step, only u_k is updated from u_k^* to u_k^{**}. The lack of hyperbolicity is handled with a relaxation approach, see

[4, 5], introducing the system $(\mathscr{S}_a^r)$ which relaxes towards $(\mathscr{S}_a)$ in the limit $\varepsilon \to 0$:

$$\begin{cases} \partial_t h_k = 0,\ \partial_t m_k = 0,\ k = 1,2, \\ \partial_t m_1 u_1 + h_1 \partial_x \Pi_I = 0, \\ \partial_t m_2 u_2 + h_2 \partial_x \Pi_2 + (\Pi_2 - \Pi_I)\partial_x h_2 = 0, \\ \partial_t m_k \Pi_k + a_k^2 h_k \partial_x u_k + a_k^2 (u_k - u_2)\partial_x h_k = \dfrac{1}{\varepsilon} m_k (\Pi_k - P_k),\ k = 1,2, \end{cases} \tag{$\mathscr{S}_a^r$}$$

where $\Pi_I = \Pi_1 - \rho_1 g \frac{h_1}{2}$ and Π_k relaxing toward P_k as $\varepsilon \to 0$. The PDE verified by Π_k is derived from the PDE verified by P_k in $(\mathscr{S})$. In addition, a_k are positive numerical parameters used to ensure the stability of the relaxation approximation in the regime of small ε, their definition is provided later according to the flow regime.

Proposition 3 (Hyperbolicity of $(\mathscr{S}_a^r)$). *When $a_k > 0$, the convective part of $(\mathscr{S}_a^r)$ is strictly hyperbolic. Its eigenvalues are given by* $\{0; \pm\frac{a_1}{\rho_1}; \pm\frac{a_2}{\rho_2}\}$.

In order to keep a numerical diffusivity based on the *material* CFL condition (7), an implicit-explicit time discretization is proposed for the convective part of $(\mathscr{S}_a^r)$:

$$\begin{cases} h_k^{**} = h_k^*, m_k^{**} = m_k^*,\ k = 1,2, \\ (m_1^{**} u_1^{**} - m_1^* u_1^*)/\Delta t + h_1^{**} \partial_x \Pi_I^{**} = 0, \\ (m_2^{**} u_2^{**} - m_2^* u_2^*)/\Delta t + h_2^{**} \partial_x \Pi_2^{**} + (\Pi_2^* - \Pi_I^*)\partial_x h_2^* = 0, \\ (m_k^{**} \Pi_k^{**} - m_k^* \Pi_k^*)/\Delta t + a_k^{2^{**}} h_k^{**} \partial_x u_k^{**} + a_k^{2^*} (u_k^* - u_2^*)\partial_x h_k^* = 0,\ k = 1,2. \end{cases} \tag{8}$$

Classical combinations on (8) lead to the following semi-discrete equations on u_k:

$$\begin{cases} \dfrac{u_1^{**} - u_1^*}{\Delta t} - \dfrac{\Delta t}{\rho_1^*} \partial_x \Big(\dfrac{a_1^{2^*}}{\rho_1^*} \partial_x u_1^{**} \Big) = -\dfrac{1}{\rho_1^*} \partial_x P_I^* + \dfrac{\Delta t}{\rho_1^*} \partial_x \Big(\dfrac{a_1^{2^*}(u_1^* - u_2^*)}{m_1^*} \partial_x h_1^* \Big), \\ \dfrac{u_2^{**} - u_2^*}{\Delta t} - \dfrac{\Delta t}{\rho_2^*} \partial_x \Big(\dfrac{a_2^{2^*}}{\rho_2^*} \partial_x u_2^{**} \Big) = -\dfrac{1}{\rho_2^*} \partial_x P_2^* - \dfrac{(P_2^* - P_I^*)}{m_2^*} \partial_x h_2^*. \end{cases} \tag{9}$$

In (9), instantaneous relaxation ($\varepsilon \to 0$) is assumed between Π_k and P_k such that $\Pi_k^* = P_k^*$. Thus, the proposed implicit relaxation approach acts as a stabilization process involving a diffusion term weighted by a_k.

Definition 1. Under the light of (9), a_k is defined according to the flow regime:

- In the stratified regime ($h_1 < H$): the pressure gradient $h_1 \partial_x P_I$ in $(\mathscr{S}_a)$ is seen as a source term. It accounts for variable interfacial pressure which can be interpreted as air phase pressure due to the relaxation (3) solved in the first step. Thus, a_1 is set to zero.
- In the pressurized regime ($h_1 = H$): the stabilization process is applied and a_1 must follow the so-called Whitham condition: $a_1^2 > \max\limits_{\rho_1}(\rho_1^2 c_1^2)$, see [4, 5].
- In all the regimes, a_2 follows the Whitham condition $a_2^2 > \max\limits_{\rho_2}(\rho_2^2 c_2^2)$.

After integrating (9) on a cell C_i and using centered schemes for gradients, one obtains an implicit system which may be written in matrix form:

$$A_k^* \mathbb{U}_k^{**} = \mathbb{S}_k^*, \tag{10}$$

where A_k^* is a non-singular tridiagonal matrix (M-matrix structure) and $\mathbb{S}_k^*$ corresponds to the integrated source term. Calculations are not detailed here. In practice, the diffusion coefficient $(a_k^2/\rho_k)^*_{i+\frac{1}{2}}$ is computed using an harmonic average and a threshold on h_1 is introduced to identify the flow regime.

Step 3: Implicit scheme for $(\mathscr{S}_u)$. In this step, only u_k is updated from u_k^{**} to u_k^{n+1}. The velocity relaxation source term is treated implicitly (except the λ_u coefficient) such that the following non-singular 2x2 system is obtained:

$$\begin{pmatrix} m_{1,i}^{**} + \Delta t \lambda_{u,i}^{**} & -\Delta t \lambda_{u,i}^{**} \\ -\Delta t \lambda_{u,i}^{**} & m_{2,i}^{**} + \Delta t \lambda_{u,i}^{**} \end{pmatrix} \begin{pmatrix} u_{1,i}^{n+1} \\ u_{2,i}^{n+1} \end{pmatrix} = \begin{pmatrix} (m_1 u_1)_i^{**} \\ (m_2 u_2)_i^{**} \end{pmatrix}. \tag{11}$$

This step concludes the overall scheme which ensures the positivity of heights and densities under the *material* CFL condition (7).

4 Numerical Results

In this section, the proposed scheme is denoted $\mathrm{SP_r}$ and compared with a classical Rusanov scheme applied on $(\mathscr{S})$ under an *acoustic* CFL condition.

Riemann problem for the convective part. One considers an analytical solution which contains two shocks for each phase traveling with the *fast* acoustic waves and a contact discontinuity (*slow* wave) where h_1 jumps. Without the pressure relaxation (3), note that a_1 follows the Whitham condition. Fields are displayed on figure 1 at $T = 23.10^{-5} s$ with 500 cells. A mesh refinement is also performed to check the numerical convergence of the method.

As expected, the $\mathrm{SP_r}$ scheme is accurate on the slow wave. Regarding the fast waves, it is more diffusive than Rusanov on phase 1 (the fastest) while better results are obtained on phase 2. Indeed, the optimal regime for the Rusanov scheme is on phase 1 with *acoustic* time steps. Stability and convergence towards relevant shock solutions are obtained with the expected convergence rate $\frac{1}{2}$ due to the contact discontinuity.

Dambreak. The source terms are activated and one considers the dambreak problem where the initial condition is a discontinuity on h_1 with constant density and zero speed. Regarding the water layer, the (incompressible) Saint-Venant system admits an analytical solution, see [8]. As the compressibility of water as well as the additional air layer should have a minor influence here, one expects to obtain the

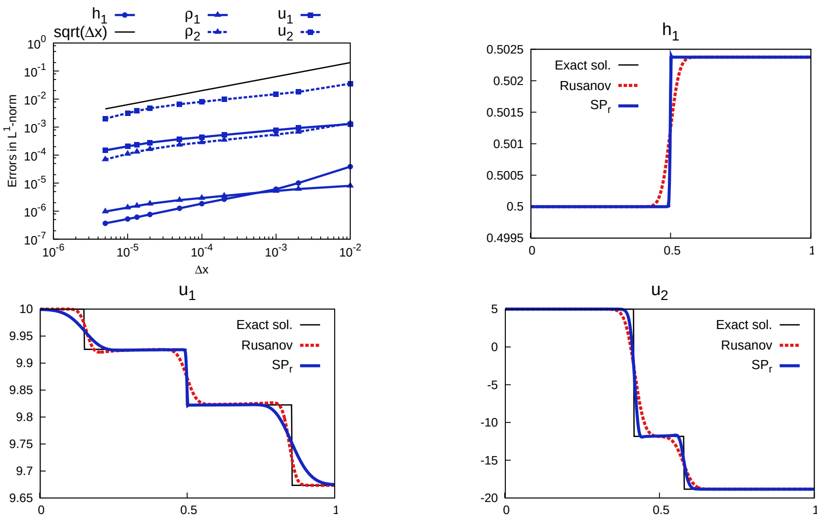

Fig. 1 Errors in L^1-norm and fields at $T = 23.10^{-5}s$ with 500 cells for the Riemann problem

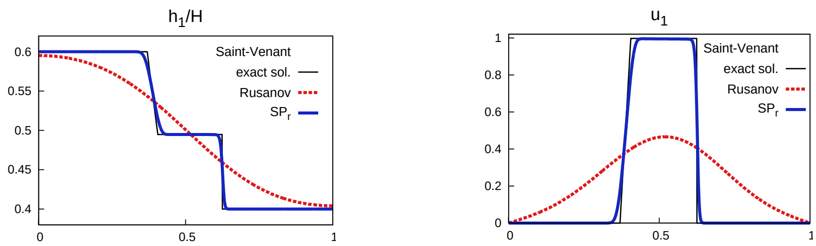

Fig. 2 Fields at $T = 24.10^{-3}s$ with 1000 cells for the dambreak problem

same kind of solution for phase 1. Therefore, a 1m long pipe is considered with $(h_1/H)_L = 0.6$ and $(h_1/H)_R = 0.4$ as initial conditions. The velocity and height fields for phase 1 are plotted on figure 2 at $T = 24.10^{-3}s$ using 1000 cells.

Contrary to the results obtained with the large time-step scheme proposed in [6], the SP_r scheme displays accurate fields regarding the Saint-Venant solution. The Rusanov scheme is highly diffusive and regarding CPU time, it needs 3 minutes while SP_r takes 6 seconds. Those results emphasize the fact that a classical explicit scheme applied on $(\mathscr{S})$ is not adapted to low speed configurations.

Mixed flow. One considers a closed sloping pipe with constant height and zero speed as initial conditions. The pipe is 5m long with $H = 1$m, $h_1 = 0.8$m, $\theta = 30$ degrees. A mesh of 250 cells is used and the threshold is set to $0.99H$. The flow becomes pressurized at the bottom (only water) and dried at the top (only air), see figure 3 for a snapshot of the water height and figure 4 for the pressure field.

Interesting qualitative results are obtained which demonstrates the ability of the

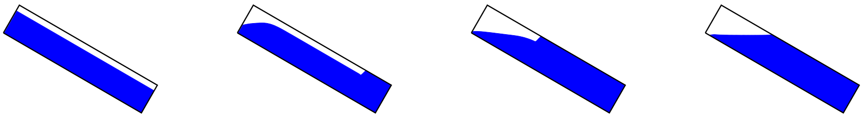

Fig. 3 Pipe filling snapshots for water height with 250 cells

SP_r scheme to handle mixed flows. Regarding the pressure field, one observes oscillations at the transition point between the regimes which are classical when dealing with mixed flows, see [1]. In the pressurized region, the pressure gradient slope is given by the expected equilibrium $\frac{\partial P_1}{\partial x} = -\rho_1 g \sin(\theta)$.

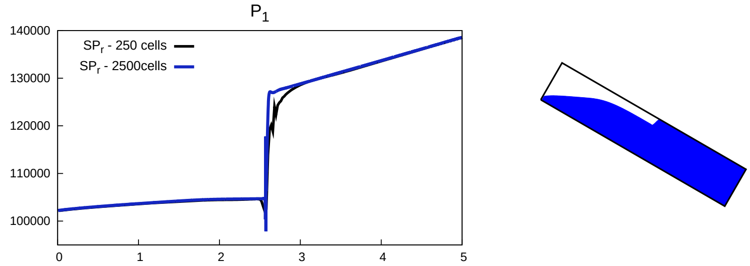

Fig. 4 Pressure field and water height at $T = 0.5s$

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References

1. Bourdarias, C., Gerbi, S.: A finite volume scheme for a model coupling free surface and pressurised flows in pipes. J. Comput. Appl. Math. **209**, 1–47 (2007)
2. Chalons, C., Coquel, F., Kokh, S., Spillane, N.: Large time-step numerical scheme for the seven-equation model of compressible two-phase flows. Springer Proceedings in Mathematics and Statistics **4**, 225–233 (2011)
3. Coquel, F., Gallouët, T., Hérard, J.M., Seguin, N.: Closure laws for a two-fluid two-pressure model. C. R. Acad. Sci. Paris **334**(I), 927–932 (2002)
4. Coquel, F., Godlewski, E., Seguin, N.: Relaxation of fluid systems. Math. Models Methods Appl. Sci. **22**(8) (2012)
5. Coquel, F., Hérard, J.M., Saleh, K.: A splitting method for the isentropic Baer-Nunziato two-phase flow model. ESAIM: Proceedings **38**(3), 241–256 (2012)
6. Demay, C., Bourdarias, C., de Laage de Meux, B., Gerbi, S., Hérard, J.M.: Numerical simulation of a compressible two-layer model: a first attempt with an implicit-explicit splitting scheme. *Submitted* (2016). URL https://hal.archives-ouvertes.fr/hal-01421889
7. Demay, C., Hérard, J.M.: A compressible two-layer model for transient gas-liquid flows in pipes. Continuum Mech. Thermodyn. **29**(2), 385–410 (2017)
8. Gerbeau, J.F., Perthame, B.: Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. Discrete Contin. Dyn. Syst. Ser. B **1**, 89–102 (2001)